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Particular solutions of the equation for the quark propagator in the infrared region of QCD

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Abstract

A number of particular solutions of the Schwinger–Dyson equation for the quark propagator in the infrared region of QCD is obtained taking account of the gauge identities. The solutions are of a non-perturbative character, some of them break the chiral invariance. The obtained solutions are analysed with the help of the effective potential method.

1. Much attention has recently been paid to the investigation of the QCD infrared region. The inapplicability of the perturbation theory in this region compels to use non-perturbative methods, in particular, the approach based on the asymptotic solution of the Schwinger–Dyson equations with regard to gauge identities [1, 2, 3, 4, 5]. The result pointing a possibility to realize the singular infrared gluon propagator asymptotics $D(k) \sim M^2/k^4$ is underlying in this approach. This result follows from the consideration of the Schwinger–Dyson equation for the gluon propagator [1, 2, 4]. The compatibility of such an asymptotics with this equation was proved in axial gauge [2, 4, 6], while in covariant gauge the problem was considered in early paper [1] in the framework of a number of approximations and treated anew recently in paper [7], where the most singular in the infrared region two-loop terms were examined for the first time. It was obtained that the infrared gluon propagator asymptotics (here and in what follows the colour indices are omitted because this does not cause any misunderstanding)

$$D_{\mu\nu}(k) = \frac{M^2}{(k^2)^2} \left(g_{\mu\nu} - d\, \frac{k_\mu k_\nu}{k^2} \right) \tag{1}$$

and the corresponding asymptotics of other lower gluon and ghost Green's functions can be described in a self-consistent way for a particular value of the gauge parameter, which is equal in the framework of dimensional regularization to

$$d = \frac{4}{5-n}\,, \tag{2}$$

where n is the space-time dimension. The distinguishing properties of this gauge were mentioned earlier when the quark propagator and the quark-gluon vertex were investigated [5, 8, 9]. It is important to emphasize that just for the value (2) free ghost Green's functions satisfy the Schwinger–Dyson equation for the ghost propagator. This fact allows us to formulate the assumption that the ghost contributions to the Slavnov–Taylor gauge identities [10] are negligible in the infrared region in this gauge. Then, in particular, the known Ward–Fradkin–Takahashi identity [11] is valid for the quark propagator $G(p)$ and the quark-gluon vertex $\Gamma_\mu(p,q;k)$:

$$k_\mu\, \Gamma_\mu(p,q;k) = G^{-1}(q) - G^{-1}(p), \qquad k = q - p\,. \tag{3}$$

In the situation when we are wittingly sure of the inapplicability of the perturbation theory, the exact solutions of the equations for Green's functions are of great significance. In the present paper we will consider the Schwinger–Dyson equation for the quark propagator, using the information contained in identity (3) for the construction of the longitudinal part of the vertex. The approximation will consist in replacing the gluon propagator by its infrared asymptotics (1) in the gauge (2).

Similar problems, in particular, in other gauges were considered in a number of papers [3, 12, 13, 14]; however, the gauge (2) with consideration of the above arguments has, from our point of view, obvious advantages. We will search for the exact solutions of the problem posed this way.

2. Consider the Schwinger–Dyson equation for the quark propagator $G(p)$:

$$1 = (\not p - m_0)\, G(p) + \frac{g^2 C_F}{(2\pi)^n \mathrm{i}} \int \mathrm{d}^n k\, D_{\mu\nu}(p-k)\, \gamma_\nu\, G(k)\, \Gamma_\mu(k,p;p-k)\, G(p)\,, \tag{4}$$

where m_0 is the bare quark mass; C_F is the factor appearing as a result of summing over the colour indices. It is equal to $(N^2-1)/(2N)$ for the group $\mathrm{SU}_c(N)$ and to 4/3 for $\mathrm{SU}_c(3)$. We use the dimensional regularization [15] for the evaluation of loop integrals (the space-time dimension is $n = 4+2\varepsilon$, $\varepsilon \to 0$). According to the initial preconditions we will use expression (1) in the gauge (2) instead of the gluon propagator $D_{\mu\nu}$. Equation (4) also contains the one-particle irreducible quark-gluon vertex Γ_μ. We will use gauge identity (3) as an additional condition imposed on this vertex. This identity permits to restore the vertex Γ_μ except for its transverse (with respect to the momentum k_μ) part

$$\Gamma_\mu(p,q;k) = \Gamma_\mu^{(\mathrm{L})}(p,q;k) + \Gamma_\mu^{(\mathrm{TR})}(p,q;k) \; ; \qquad q-p-k=0 \; . \tag{5}$$

It should be mentioned here that the division of the vertex into the longitudinal, $\Gamma_\mu^{(\mathrm{L})}$, and transverse, $\Gamma_\mu^{(\mathrm{TR})}$, parts is somewhat arbitrary. To define the longitudinal part restored with identity (3) from the quark propagator form, we will apply a convenient representation equivalent to that obtained in Ref. [16]:

$$\Gamma_\mu^{(\mathrm{L})}(p,q;q-p) = \frac{1}{q^2-p^2}\left[(\not p\gamma_\mu + \gamma_\mu \not q)G^{-1}(q) - G^{-1}(p)(\not p\gamma_\mu + \gamma_\mu \not q)\right] \; . \tag{6}$$

Further, if the quark propagator is presented as

$$G(p) = \widetilde{A}(p^2)\not p + \widetilde{B}(p^2) = \widetilde{A}(p^2)(\not p + \mu(p^2)) \; , \tag{7}$$

then, considering the loop integral in Eq. (4) (cf. Ref. [17]), one can see that the transverse part of the vertex allows the following functional arbitrariness:

$$\begin{aligned}\Gamma_\mu^{(\mathrm{F})}(p,q;q-p) &= 2F(p,q)(\not p + \mu(p^2))^{-1}\,(\not p\gamma_\mu \not q - \not q\gamma_\mu \not p)\,(\not q + \mu(q^2))^{-1} \\ &= 2\widetilde{F}(p,q)\; G^{-1}(p)\;\; (\not p\gamma_\mu \not q - \not q\gamma_\mu \not p)\;\; G^{-1}(q) \; ,\end{aligned} \tag{8}$$

where $\widetilde{F}(p,q) = \widetilde{A}(p^2)\widetilde{A}(q^2)F(p,q)$; $F(p,q)$ is an arbitrary scalar symmetric function.

When studying the solutions of Eq. (4) we will take into account the above arbitrariness. However, since it does not influence the initial equation (4), we can represent the transverse part of the vertex in the following form:

$$\Gamma_\mu^{(\mathrm{TR})}(p,q;q-p) = \Gamma_\mu^{(\mathrm{T})}(p,q;q-p) + \Gamma_\mu^{(\mathrm{F})}(p,q;q-p) \; , \tag{9}$$

where the form of $\Gamma_\mu^{(\mathrm{T})}(p,q;q-p)$ is fixed by the requirement that it should satisfy Eq. (4), while $\Gamma_\mu^{(\mathrm{F})}(p,q;q-p)$ is defined by formula (8). As for $\Gamma_\mu^{(\mathrm{T})}$, this part of the vertex is assumed to contain the same transverse (with respect to $(q-p)_\mu$) structure $(\not p\gamma_\mu \not q - \not q\gamma_\mu \not p)$, which is between two matrix structures of a more general form

$$\Gamma_\mu^{(\mathrm{T})}(p,q;q-p) = 2\left[K(p^2)(\not p + m_0) + L(p^2)\right]^{-1}(\not p\gamma_\mu \not q - \not q\gamma_\mu \not p)\left[K(q^2)(\not q + m_0) + L(q^2)\right]^{-1} . \tag{10}$$

Note that (10) is not the most general possible expression for $\Gamma_\mu^{(\mathrm{T})}$. For example, this expression could be multiplied by an arbitrary scalar symmetric function $f(p,q)$, which cannot be represented as $\psi(p^2)\psi(q^2)$. However, the choice of the transverse part (10) of the vertex allows us to obtain a number of explicit solutions of Eq. (4).

It will be convenient to present the quark propagator as

$$G(p) = \left(A(p^2) + B(p^2)\frac{\not{p}}{p^2} \right) \left[K(p^2)(\not{p} + m_0) + L(p^2) \right] . \tag{11}$$

Then the scalar functions $\widetilde{A}$ and $\widetilde{B}$ of the matrix structures of the quark propagator (7) are equal to

$$\left.\begin{aligned} \widetilde{A}(p^2) &= A(p^2)K(p^2) + \frac{1}{p^2}B(p^2)(m_0K(p^2) + L(p^2)) , \\ \widetilde{B}(p^2) &= A(p^2)(m_0K(p^2) + L(p^2)) + B(p^2)K(p^2) . \end{aligned}\right\} \tag{12}$$

To evaluate the integral in Eq. (4) it is convenient to use the formula

$$\int \mathrm{d}^n k \; D_{\mu\nu}(p-k) \; \gamma_\mu \left[\not{k} f_1(k^2) + f_2(k^2) \right] \gamma_\nu = 6\mathrm{i}\pi^2 M^2 \left[\not{p} f_1(p^2) - f_2(p^2) \right] , \tag{13}$$

where $D_{\mu\nu}$ is the gluon propagator (1) in the gauge (2), f_1 and f_2 are arbitrary scalar functions, whose Fourier transforms in configuration space do not vanish (in the framework of dimensional regularization) and do not contain the singularities with respect to $\varepsilon = (n-4)/2$. On the r.h.s. of equality (13) the terms $\sim \mathcal{O}(\varepsilon)$ vanishing with dimensional regularization removal are omitted. The validity of formula (13) under the conditions mentioned can be proved by means of, for example, transition into the x-space, where in the gauge (2) the gluon propagator is transverse.

Putting expressions (1), (6), (10), (11) into (4) we obtain

$$\begin{aligned} 1 &= \frac{\not{p}}{p^2} \left[p^2 A(p^2)L(p^2) + (p^2 - m_0^2)B(p^2)K(p^2) - m_0 B(p^2)L(p^2) \right] \\ &\quad + \left[(p^2 - m_0^2)A(p^2)K(p^2) - m_0 A(p^2)L(p^2) + B(p^2)L(p^2) \right] \\ &\quad + \frac{g^2\, C_F}{(2\pi)^4 \mathrm{i}} \left(I^{(\mathrm{L})}(p) + I^{(\mathrm{T})}(p) \right) , \end{aligned} \tag{14}$$

where $I^{(\mathrm{L})}(p)$ and $I^{(\mathrm{T})}(p)$ correspond to the contributions to the loop integral from the items with the longitudinal (6) and transverse (10) parts of the vertex, respectively. Using formula (13) to evaluate $I^{(\mathrm{L})}(p)$, we get the following expressions (in the limit of $n \to 4$):

$$I^{(\mathrm{L})}(p) = 6\mathrm{i}\pi^2 M^2 \widetilde{A}(p^2) = 6\mathrm{i}\pi^2 M^2 \left[A(p^2)K(p^2) + \frac{1}{p^2}B(p^2)(m_0K(p^2) + L(p^2)) \right] , \tag{15}$$

$$I^{(\mathrm{T})}(p) = \frac{8M^2}{p^2} \left(A(p^2)\not{p} + B(p^2) \right) \int \mathrm{d}^4 k \; \frac{B(k^2)}{k^2((p-k)^2)^2} \left(p^2 k^2 - (p\cdot k)^2 \right) . \tag{16}$$

Putting (15) and (16) in (14) and equating the expressions of the relevant matrix structures, we obtained the following system of equations:

$$\left.\begin{aligned} 1 &= (p^2 - m_0^2 + \kappa^2)A(p^2)K(p^2) + \frac{p^2 + \kappa^2}{p^2}B(p^2)L(p^2) + \frac{m_0\kappa^2}{p^2}B(p^2)K(p^2) \\ &\qquad - m_0 A(p^2)L(p^2) + \frac{\kappa^2}{p^2}B(p^2)I(p^2|B), \\ 0 &= A(p^2)L(p^2) + \frac{p^2 - m_0^2}{p^2}B(p^2)K(p^2) - \frac{m_0^2}{p^2}B(p^2)L(p^2) + \frac{\kappa^2}{p^2}A(p^2)I(p^2|B). \end{aligned}\right\} \tag{17}$$

We have introduced here the notations

$$I(p^2|B) \;=\; \frac{4}{3\mathrm{i}\pi^2}\int \mathrm{d}^4k\, \frac{B(k^2)}{k^2((p-k)^2)^2}\left(p^2k^2-(p\cdot k)^2\right)\,, \tag{18}$$

$$\kappa^2 \;=\; \frac{3g^2M^2C_F}{8\pi^2} \qquad \left(\text{for } \mathrm{SU}_c(3), \quad \kappa^2=\frac{g^2M^2}{2\pi^2}\right)\,. \tag{19}$$

Note that the value of κ^2 can be related to the slope parameter of the linear part of the quark-antiquark experimentally determined potential.

To investigate system (17) we will have to analyse the equations containing the integral operator (18). It will be convenient to perform this analysis in the four-dimensional Euclidean space-time, as has been done in Ref. [18]. Then, introducing the notations $x=p_{\mathrm{E}}^2=-p^2$, $y=k_{\mathrm{E}}^2=-k^2$ and keeping the notations for the investigated functions the same and performing angular integration, we obtain the following expression for the integral (18):

$$I(x|B)=-\frac{1}{x}\int\limits_0^x y\,\mathrm{d}y\,B(y)-x\int\limits_x^\infty\frac{\mathrm{d}y}{y}\,B(y)\,. \tag{20}$$

In some cases it would be convenient to pass from integral equations to differential ones. For the equation containing the integral operator (20) and an arbitrary r.h.s. $f(x)$,

$$\frac{1}{x}\int\limits_0^x y\,\mathrm{d}y\,B(y)+x\int\limits_x^\infty\frac{\mathrm{d}y}{y}\,B(y)=f(x)\,, \tag{21}$$

we obtain the following second-order differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^3\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{x}\right)\right]=-2xB(x)\,. \tag{22}$$

Equation (22) is equivalent to (21) under the following boundary conditions:

$$\lim_{y\to 0}\left[y^3\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{f(y)}{y}\right)\right]=0\,, \qquad \lim_{y\to\infty}\left[\frac{1}{y}\frac{\mathrm{d}}{\mathrm{d}y}\left(yf(y)\right)\right]=0\,. \tag{23}$$

Consider then the particular cases of system (17). If we put $B(p^2)=0$, then from Eqs. (17) we obtain

$$\left.\begin{aligned} &1=(p^2-m_0^2+\kappa^2)A(p^2)K(p^2)\,,\\ &0=A(p^2)L(p^2)\,. \end{aligned}\right\}$$

As a result, we get a particular solution to the Schwinger–Dyson equation found in Ref. [17],

$$\left.\begin{aligned} &G(p)=\frac{\not p+m_0}{p^2-m_0^2+\kappa^2}\,,\\ &\Gamma_\mu^{(\mathrm{L})}(p,q;q-p)=\gamma_\mu-\kappa^2(\not p-m_0)^{-1}\gamma_\mu(\not q-m_0)^{-1}\,,\\ &\Gamma_\mu^{(\mathrm{T})}(p,q;q-p)=0\,,\\ &\Gamma_\mu^{(\mathrm{F})}(p,q;q-p)=2F(p,q)(\not p+m_0)^{-1}\left(\not p\gamma_\mu\not q-\not q\gamma_\mu\not p\right)(\not q+m_0)^{-1}\,. \end{aligned}\right\} \tag{24}$$

To satisfy the Schwinger–Dyson equation, we did not have to introduce the transverse part $\Gamma_\mu^{(\mathrm{T})}$ into the solution above. For $m_0^2 \geq \kappa^2$, the propagator in Eq. (24) has the "right" pole corresponding to the mass $m = \sqrt{m_0^2 - \kappa^2}$, while for $m_0^2 < \kappa^2$ this pole turns out in the non-physical "tachyonic" region.

Consider then another particular case of system (17), $A(p^2) = 0$. After some transformations, it takes the following form:

$$\left.\begin{aligned} 1 &= \frac{1}{m_0}(p^2 - m_0^2 + \kappa^2)B(p^2)K(p^2) + \frac{\kappa^2}{p^2}B(p^2)I(p^2|B)\ , \\ 0 &= (p^2 - m_0^2)K(p^2) - m_0 L(p^2)\ . \end{aligned}\right\} \tag{25}$$

A particular solution of system (25) can be obtained explicitly by putting

$$B(p^2)K(p^2) = \frac{m_0}{p^2}\ . \tag{26}$$

Then we obtain the following integral equation of the form of Eq. (21):

$$\frac{1}{x}\int_0^x y\ \mathrm{d}y\ B(y) + x\int_x^\infty \frac{\mathrm{d}y}{y}\ B(y) = \frac{\kappa^2 - m_0^2}{\kappa^2 B(x)} \tag{27}$$

and the corresponding differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^3\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{xB(x)}\right)\right] = -\frac{2\kappa^2}{\kappa^2 - m_0^2}\ xB(x)\ . \tag{28}$$

In spite of the fact that (28) is a complicated non-linear second-order differential equation, one can construct its particular power-like solution satisfying the boundary conditions (23),

$$B(x) = \beta x^{-1/2}\ , \qquad \beta^2 = \frac{3(\kappa^2 - m_0^2)}{8\kappa^2}\ .$$

Note that we managed to obtain the solution of (25) only by imposing condition (26). In other cases we failed to find the solution satisfying the boundary conditions. Thus, for $A(p^2) = 0$, the following particular solution of initial system (17) is obtained:

$$\left.\begin{aligned} &G(p) = \frac{\not p + m_0}{p^2}, \\ &\Gamma_\mu^{(\mathrm{L})}(p, q; q-p) = \gamma_\mu - m_0^2(\not p + m_0)^{-1}\gamma_\mu(\not q + m_0)^{-1}, \\ &\Gamma_\mu^{(\mathrm{T})}(p, q; q-p) = \frac{3(\kappa^2 - m_0^2)}{4\kappa^2}\frac{\not p}{\sqrt{-p^2}}(\not p + m_0)^{-1}(\not p\gamma_\mu\not q - \not q\gamma_\mu\not p)(\not q + m_0)^{-1}\frac{\not q}{\sqrt{-q^2}}, \\ &\Gamma_\mu^{(\mathrm{F})}(p, q; q-p) = 2F(p,q)(\not p + m_0)^{-1}(\not p\gamma_\mu\not q - \not q\gamma_\mu\not p)(\not q + m_0)^{-1}. \end{aligned}\right\} \tag{29}$$

In this case, to satisfy the Schwinger–Dyson equation, we have to introduce the non-trivial transverse part $\Gamma_\mu^{(\mathrm{T})}$ of the vertex. For any m_0 the propagator in (29) has a simple pole corresponding to the zero mass. For $m_0 = 0$ solution (29), like (24), is chiral-invariant. It is also interesting to note that for $m_0 = \kappa$ solutions (24) and (29) coincide.

We will consider now the case in which $B(p^2) \neq 0$ and $A(p^2) \neq 0$. Then, extracting the integral $I(p^2|B)$ from the second equation of (17) and putting it into the first one, we obtain another system. One of the equations is algebraic, and the second one remains integral,

$$\left.\begin{aligned}1 &= (p^2 - m_0^2 + \kappa^2)A(p^2)K(p^2) - m_0 A(p^2)L(p^2) + \frac{m_0\kappa^2}{p^2}B(p^2)K(p^2) \\ &\quad + \frac{\kappa^2}{p^2}B(p^2)L(p^2) - \frac{(p^2-m_0^2)}{p^2}\frac{B^2(p^2)K(p^2)}{A(p^2)} + \frac{m_0}{p^2}\frac{B^2(p^2)L(p^2)}{A(p^2)}, \\ \kappa^2 I(p^2|B) &= -p^2 L(p^2) - (p^2 - m_0^2)\frac{B(p^2)K(p^2)}{A(p^2)} + m_0 B(p^2)L(p^2).\end{aligned}\right\} \tag{30}$$

To obtain a particular solution of this system we introduce additional assumptions

$$B(p^2) = \lambda A(p^2), \qquad L(p^2) = \nu K(p^2), \qquad \lambda, \nu = \text{const.} \tag{31}$$

Putting the constraints (31) in Eqs. (30), we arrive at the following system:

$$\left.\begin{aligned}1 &= A(p^2)K(p^2)\left[p^2 - m_0^2 + \kappa^2 - m_0\nu - \lambda^2 + \frac{\lambda}{p^2}(m_0+\nu)(\kappa^2 + \lambda m_0)\right], \\ \kappa^2 I(p^2|B) &= K(p^2)\left[-(\nu+\lambda)p^2 + m_0\lambda(m_0+\nu)\right].\end{aligned}\right\} \tag{32}$$

One can see from here that, with the conditions

$$\left.\begin{aligned}&\lambda(m_0+\nu)(\kappa^2+\lambda m_0) = 0, \\ &(\kappa^2 - \lambda^2)(\nu+\lambda) = m_0\nu(m_0+\nu), \qquad \nu + \lambda \neq 0,\end{aligned}\right\} \tag{33}$$

satisied, we may reduce the problem to the differential equation, similar to Eq. (28) (with different coefficients in its r.h.s.). Its particular solution satisfying the boundary conditions has already been obtained.

Equations (33) make it possible to obtain a set of possible values of λ and ν. Note that the case when $\lambda = 0$ has already been considered, i.e. $B(p^2) = 0$.

Consider now the case $\nu = -m_0$. Then we obtain $\lambda^2 = \kappa^2$. Let us consider first $\lambda = +\kappa$ (all expressions for $\lambda = -\kappa$ will differ by the sign of κ only). The corresponding differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^3\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{xB(x)}\right)\right] = -\frac{2\kappa}{\kappa - m_0}\,xB(x)\,,$$

has the power-like solution satisfying the boundary conditions (23),

$$B(x) = \beta' x^{-1/2}\,, \qquad \beta'^2 = \frac{3(\kappa - m_0)}{8\kappa}\,.$$

The expressions for Green's functions in this case take the form

$$\left.\begin{aligned}&G(p) = \frac{\not{p} + \kappa}{p^2}, \\ &\Gamma_\mu^{(\mathrm{L})}(p,q;q-p) = \gamma_\mu - \kappa^2(\not{p}+\kappa)^{-1}\gamma_\mu(\not{q}+\kappa)^{-1}, \\ &\Gamma_\mu^{(\mathrm{T})}(p,q;q-p) = \frac{3(\kappa - m_0)}{4\kappa^3}\frac{\not{p}}{\sqrt{-p^2}}(\not{p}\gamma_\mu\not{q} - \not{q}\gamma_\mu\not{p})\frac{\not{q}}{\sqrt{-q^2}}, \\ &\Gamma_\mu^{(\mathrm{F})}(p,q;q-p) = 2F(p,q)(\not{p}+\kappa)^{-1}(\not{p}\gamma_\mu\not{q} - \not{q}\gamma_\mu\not{p})(\not{q}+\kappa)^{-1}.\end{aligned}\right\} \tag{34}$$

Solution (34), like that of (29), has a non-trivial $\Gamma^{(\mathrm{T})}_\mu$, and the pole in the propagator also corresponds to the zero mass. However, in contrast to solutions (24) and (29), for $m_0 = 0$ solution (34) is not chiral invariant. It should also be noted that for $m_0 = \kappa$ solution (34) coincides with solutions (24) and (29) obtained earlier. If we change the sign of κ in solution (34), then we obtain the following expressions for the propagator and the vertex:

$$\left.\begin{aligned}
&G(p) = \frac{\not p - \kappa}{p^2},\\
&\Gamma^{(\mathrm{L})}_\mu(p,q;q-p) = \gamma_\mu - \kappa^2(\not p - \kappa)^{-1}\gamma_\mu(\not q - \kappa)^{-1},\\
&\Gamma^{(\mathrm{T})}_\mu(p,q;q-p) = \frac{3(\kappa + m_0)}{4\kappa^3}\frac{\not p}{\sqrt{-p^2}}(\not p\gamma_\mu\not q - \not q\gamma_\mu\not p)\frac{\not q}{\sqrt{-q^2}},\\
&\Gamma^{(\mathrm{F})}_\mu(p,q;q-p) = 2F(p,q)(\not p - \kappa)^{-1}(\not p\gamma_\mu\not q - \not q\gamma_\mu\not p)(\not q - \kappa)^{-1}.
\end{aligned}\right\} \tag{35}$$

The last possibility to satisfy the first condition in (33) consists in putting $\lambda = -\kappa^2/m_0$. Then it follows from (33) that either $\nu = (\kappa^2 - m_0^2)/m_0$ or $\nu = -\kappa^4/m_0^3$.

Let us consider now the first case, i.e., $\lambda = -\kappa^2/m_0$, $\nu = (\kappa^2 - m_0^2)/m_0$. Then the power-like solution of the corresponding differential equation satisfying the boundary conditions (23) is of the form

$$B(x) = \beta'' x^{-1/2}\,, \qquad \beta''^2 = \tfrac{3}{8}\,.$$

Going over to the expressions for Green's functions we get

$$\left.\begin{aligned}
&G(p) = \frac{\not p}{p^2},\\
&\Gamma^{(\mathrm{L})}_\mu(p,q;q-p) = \gamma_\mu,\\
&\Gamma^{(\mathrm{T})}_\mu(p,q;q-p) = \frac{3}{4\kappa^4}\frac{(m_0\not p - \kappa^2)}{\sqrt{-p^2}}(\not p\gamma_\mu\not q - \not q\gamma_\mu\not p)\frac{(m_0\not q - \kappa^2)}{\sqrt{-q^2}},\\
&\Gamma^{(\mathrm{F})}_\mu(p,q;q-p) = 2F(p,q)(\not p)^{-1}(\not p\gamma_\mu\not q - \not q\gamma_\mu\not p)(\not q)^{-1}.
\end{aligned}\right\} \tag{36}$$

A distinguishing feature of this solution is that for any value of m_0 the propagator (36) obeys the free Dirac equation with zero mass. This solution also has a non-trivial $\Gamma^{(\mathrm{T})}_\mu$, as in the case with (29) and (34). Note that the presence of κ in the denominators of the expressions for $\Gamma^{(\mathrm{T})}_\mu$ in (29), (34), (36) points directly to the non-perturbative nature of these solutions. Besides, for $m_0 = 0$ solution (36) coincides with chiral-invariant solution (29), but for $m_0 = \kappa$ it does not coincide with the solutions obtained earlier.

It remains to consider the case $\lambda = -\kappa^2/m_0^2$, $\nu = -\kappa^4/m_0^3$. The differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^3\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{xB(x)}\right)\right] = -\frac{2m_0^4}{\kappa^2(\kappa^2 + m_0^2)}\,xB(x)$$

has in this case the following particular solution satisfying the boundary conditions (23):

$$B(x) = \beta''' x^{-1/2}\,, \qquad \beta'''^2 = \frac{3\kappa^2(\kappa^2 + m_0^2)}{8m_0^4}\,.$$

The construction of Green's functions leads to the expressions

$$\left.\begin{aligned} G(p) &= \frac{\kappa^2(\kappa^2+m_0^2)}{m_0^4}\frac{\not p}{p^2} - \frac{\kappa^4+\kappa^2 m_0^2 - m_0^4}{m_0^4}\frac{\not p + m_0}{p^2 - m_0^2+\kappa^2}, \\ \Gamma_\mu^{(\mathrm{L})}(p,q;q-p) &= \gamma_\mu + \kappa^2\frac{\kappa^4+\kappa^2m_0^2-m_0^4}{\kappa^4-\kappa^2m_0^2-m_0^4}[m_0^2(m_0\not p-\kappa^2)^{-1}\gamma_\mu(m_0\not q-\kappa^2)^{-1} \\ &\quad -(\kappa^2-m_0^2)(\kappa^4-m_0^4)(m_0^3\not p-\kappa^4+m_0^4)^{-1}\gamma_\mu(m_0^3\not q-\kappa^4+m_0^4)^{-1}], \\ \Gamma_\mu^{(\mathrm{T})}(p,q;q-p) &= \frac{3m_0^4(\kappa^2+m_0^2)}{4\kappa^2}\frac{(p^2+\kappa^2-m_0^2)}{\sqrt{-p^2}}(m_0^3\not p-\kappa^4+m_0^4)^{-1} \\ &\quad \times(\not p\gamma_\mu\not q-\not q\gamma_\mu\not p)(m_0^3\not q-\kappa^4+m_0^4)^{-1}\frac{(q^2+\kappa^2-m_0^2)}{\sqrt{-q^2}}, \end{aligned}\right\} \quad (37)$$

where the corresponding functional arbitrariness should be added. The propagator in (37) is a linear combination of the propagators of (36) and (24), the coefficients being singular as $m_0 \to 0$. Note also that for $m_0 = \kappa$ both solutions (35) and (37) coincide, for $m_0 = \kappa\sqrt{(1+\sqrt{5})/2}$ solution (37) coincides with solution (36), and at $m_0 \to \infty$ solution (37) transforms into (24).

3. Thus we have obtained six different solutions for our equation. The propagators in two of them, (24) and (37), have tachyonic singularities for $m_0 < \kappa$, and those in four other solutions, (29), (34), (35), (36), have poles in the origin in the p^2-plane which correspond to the zero mass. We cannot claim that we have found all the solutions possible, because the solutions of Eq. (22) are arbitrary to a high extent due to arbitrariness of the transverse part of the vertex (functions $K(p^2)$ and $L(p^2)$ in our approach). However, we should not forget that it is necessary to satisfy both the boundary conditions (23) and the requirement that Green's functions should have correct analytic properties. The latter condition, for example, limits the domain where solutions (24) and (37) are defined, to $m_0 \geq \kappa$. The attempts to find other solutions in some particular cases led to the violation of one of the above conditions.

A characteristic feature of the solutions obtained is their fairly simple algebraic form which, in particular, makes the analysis of their properties easier. Note also that each of the solutions depends parametrically on m_0 and has a certain functional arbitrariness in the transverse part of the vertex.

It is interesting to consider the values of m_0 for which different solutions coincide. As already mentioned, for $m_0 = \kappa$ solutions (24), (29) and (34) coincide. Keeping in mind that solution (24) is not defined for $m_0 < \kappa$, we get a qualitative picture of the "splitting" of the solutions with m_0 decreasing. Really, let massive solution (24) be realized when $m_0 > \kappa$. Then for $m_0 = \kappa$ it can be transformed continuously into either of the two massless solutions, (29) and (34), the latter breaking chiral inveriance for $m_0 = 0$ and the former conserving it.

A similar picture is observed for three other solutions, (35), (36), (37), the only difference in this case the transition from solution (37) into (35) and (36) taking place at different points, at $m_0 = \kappa$ into (35) and at $m_0 = \kappa\sqrt{(1+\sqrt{5})/2}$ into (36). One of these solutions, (35), breaks chiral invarince.

Whenever there are a number of solutions, the problem of choice arises: which of them is preferable. For our problem, the method of the effective potential for composite operators [19] is convenient for comparing the solutions. This method is used intensively nowadays to investigate the dynamic chiral symmetry breaking in QCD (see, for example, Ref. [20]). To compare the energetic properties of our solutions, we have to calculate the evvective potential

$$V(G) = \frac{1}{(2\pi)^4 \mathrm{i}} \int \mathrm{d}^4p \; \mathrm{Tr} \left[\ln(G_0^{-1}G) - G_0^{-1}G + 1 \right] + V^{(2)}(G), \tag{38}$$

where $V^{(2)}(G)$ is the sum of all two-particle irreducible vacuum diagrams (Fig. 1). The pure gluonic contributions to the effective potential are the same for all the solutions, therefore we include them into the normalization $V(G_0) = 0$. In the framework of our approach one can calculate $V^{(2)}(G)$ and, hence, $V(G)$ exactly. Really, the condition that the functional (38) should have an extremum, i.e. $\delta V/\delta G = 0$, is just the Schwinger–Dyson equation for the quark propagator (4) treated above. For its solutions, i.e., when the condition $\delta V/\delta G = 0$ is satisfied, it is easy to get a simple expression for $V^{(2)}(G)$ instead of the two-loop integral of the diagram in Fig. 1,

$$V^{(2)}(G)\Big|_{\delta V/\delta G=0} = \frac{1}{2\mathrm{i}(2\pi)^4} \int \mathrm{d}^4p \; \mathrm{Tr} \left[G_0^{-1}G - 1 \right] .$$

As a result, exression (38) takes the form

$$V(G) = \frac{1}{2\mathrm{i}(2\pi)^4} \int \mathrm{d}^4p \; \mathrm{Tr} \left[2\ln(G_0^{-1}G) - G_0^{-1}G + 1 \right] . \tag{39}$$

Let us emphasize once more that formula (39) yields the exact value of the effective potential for extrema, i.e. for the solutions of Eq. (4) which have been obtained earlier.

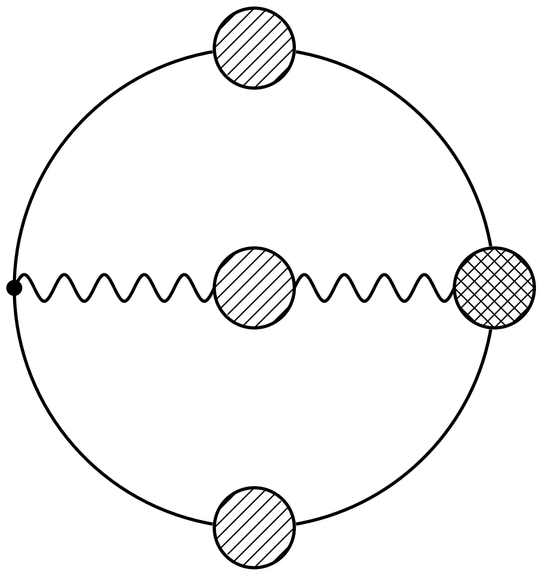

Figure 1: The diagram for $V^{(2)}(G)$.

Having explicit expressions of the propagators G, we can calculate the functional (39). However, in doing that, we face ultraviolet divergences. It should be kept in mind that the solutions obtained can be correct in the infrared region only, i.e. for $p_{\mathrm{E}}^2 < \Lambda^2$, where

Λ is an effective cutoff confining the infrared region. One can expect the value of the cutoff Λ in (39) to be of the order of parameter k_0 introduced into the theory as the characteristic momentum separating the infrared and ultraviolet regions. This parameter has been defined, in particular, in papers [21] and established as $k_0 \sim 0.7$ GeV.

Following these arguments we will take $\Lambda \sim k_0$ in the subsequent calculations of the effective potential.

Thus, substituting the expressions for G into (39) and carrying out calculations of the integrals in the four-dimensional Euclidean space with the cutoff Λ as the upper limit, we get expressions for the effective potential for our solutions. For example, for solution (24) we obtain

$$V_1 = \frac{N}{8\pi^2}\left\{\Lambda^4 \ln\frac{\Lambda^2+m_0^2}{\Lambda^2+m_0^2-\kappa^2} - m_0^4 \ln\frac{\Lambda^2+m_0^2}{m_0^2} + m_0^2(m_0^2-\kappa^2)\ln\frac{\Lambda^2+m_0^2-\kappa^2}{m_0^2-\kappa^2}\right\}. \quad (40)$$

The potential V_2 of solution (29) follows from Eq. (40) under substitution $\kappa = m_0$. For solution (34) we get

$$V_3 = \frac{N}{8\pi^2}\left\{\frac{1}{2}\Lambda^4 \ln\frac{\Lambda^2+m_0^2}{\Lambda^2} + \frac{1}{2}\Lambda^4 \ln\frac{\Lambda^2+\kappa^2}{\Lambda^2} - \frac{1}{2}m_0^4 \ln\frac{\Lambda^2+m_0^2}{m_0^2} - \frac{1}{2}\kappa^4 \ln\frac{\Lambda^2+\kappa^2}{\kappa^2} + \frac{(\kappa-m_0)^2}{2}\Lambda^2\right\}. \quad (41)$$

The potential V_4 for solution (35) is obtained from Eq. (41) with $\kappa \to -\kappa$, and V_5 corresponding to (36) by substitution $\kappa = 0$. We do not give here the expression for the potential V_6 (solution (37)) because it is cumbersome.

The behaviour of the effective potential as a function of m_0 for our solutions is presented in Fig. 2 at fixed $\kappa/\Lambda = 0.6$. The value of $\kappa = gM/(\pi\sqrt{2})$ (for $N = 3$) can be related to the slope parameter of the linear part of the quark-antiquark potential and is usually taken equal to 0.42 GeV. Thus, Fig. 2 corresponds to the case when $\Lambda = 0.7$ GeV. We see that for massive solutions (24) and (37) the effective potentials V_1 and V_6 are positive for every $m_0 \geq \kappa$ and tend to zero for large m_0. The effective potential V_5 of chiral invariant solution (36) increases monotonously with m_0. The potentials V_2 and V_3 have the intervals of growth and falling, and $V_2 < 0$ for $m_0 > \Lambda$, V_3 also becomes negative for large m_0, V_2 and $V_3 \to -\infty$ as $m_0 \to \infty$. Note that the potential V_3 of chiral non-invariant solution (34) has minimum at the point $m_0 \simeq 0.5\Lambda$.

It is to the point to say about our interpretation of the parameter m_0. In the framework of the problem dealing with the infrared region, m_0 is an external parameter. It includes the current quark mass $\widetilde{m}$ (which is not chromodynamic by origin) and a contribution Δm obtained from the true QCD interaction at moderate and small distances. This contribution can be large enough even in the case of a small value of $\widetilde{m}$.

Calculations for different $C = \kappa/\Lambda$ show that for $m_0 > 0$ the inequalities $V_4 > V_3$, $V_5 > V_2$ and $V_6 > V_1$ are always held. Therefore, the dynamics of the change in the effective potentials for various values of C versus m_0/Λ is illustrated in Fig. 3 for the first three solutions only.

In dimensionless variables used in Fig. 3 the function V_2 is independent of C. In the range of $\Lambda = (0.5 \div 1.0)$ GeV presenting the most interest and corresponding to $C =$

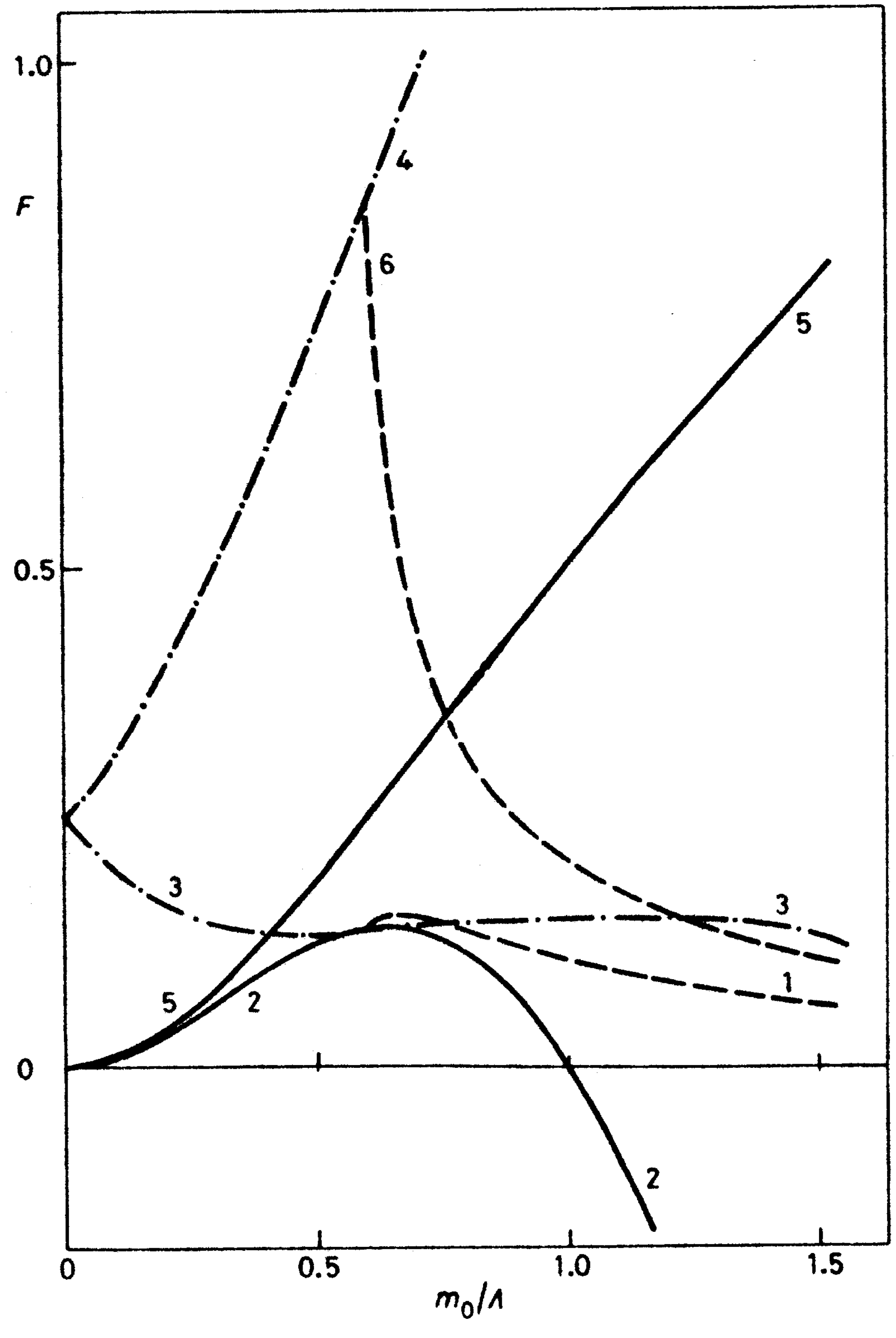

Figure 2: The plot of V_i ($i = 1, \ldots, 6$) versus m_0/Λ for $\kappa/\Lambda = 0.6$ (the quantity $F = 8\pi^2 V/(N\Lambda^4)$ is plotted).

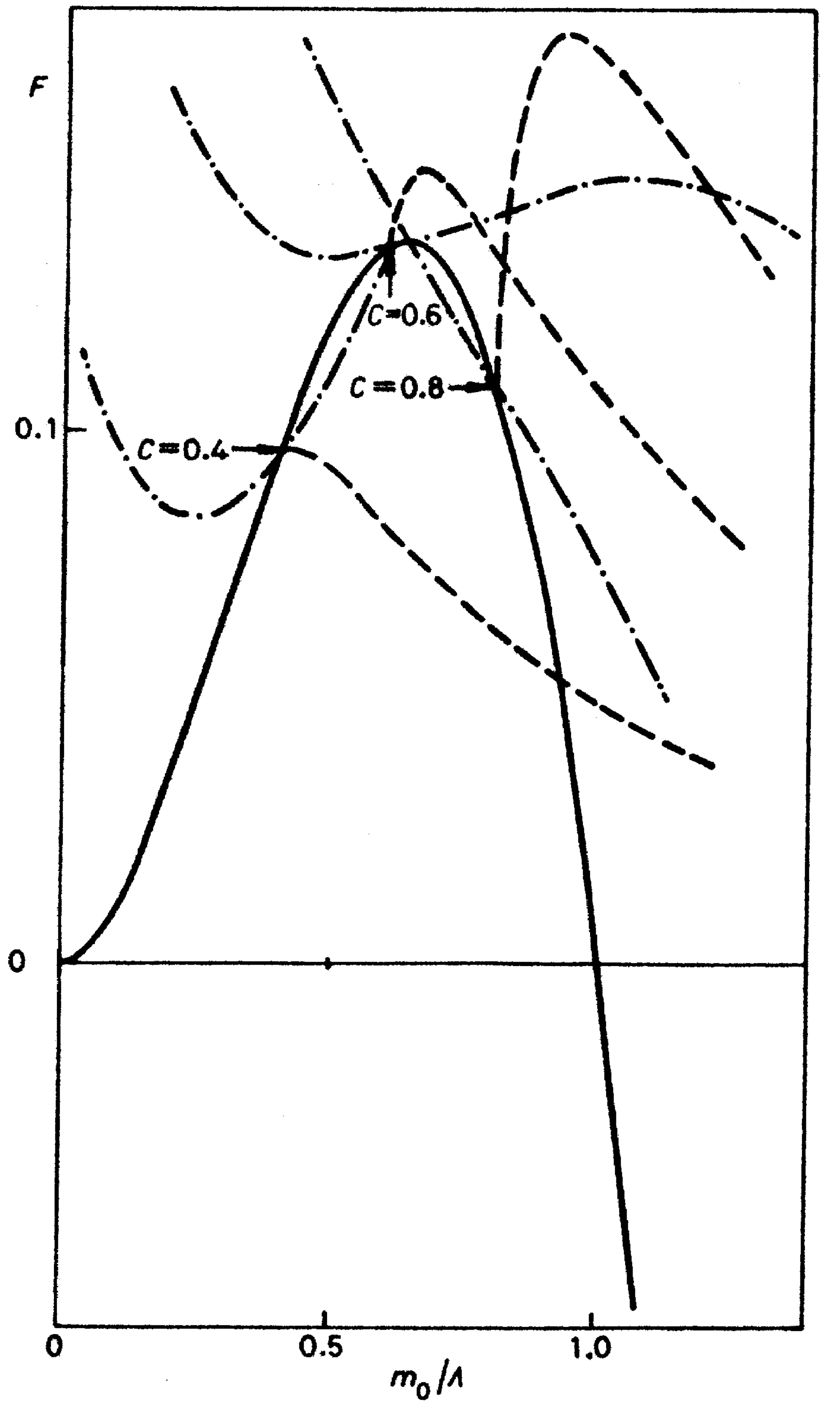

Figure 3: The behaviour of V_i ($i = 1, 2, 3$) near the branching point for various $C = \kappa/\Lambda$. (V_1 is the dashed curve, V_2 is the solid one, and V_3 is the dashed and dotted one.)

$0.8 \div 0.4$ in Fig. 3, any of these three solutions may have the minimal value of the energy density. This depends on the value of the external parameter m_0. For example, in the case $C = 0.8$ chiral-invariant solution (34) can be realized for 0.34 GeV $< m_0 <$ 0.42 GeV.

Figures 2 and 3 show a complicated picture of the phase transitions between the solutions obtained. It is of interest to note that the position of the minimum for the potential V_3 corresponding to chiral-invariance breaking solution (34) is fixed in the limit $\Lambda \to \infty$ ($C \to 0$) and is equal to $m_0 = \kappa/2 = 0.21$ GeV.

4. Thus, a number of the particular solutions for Eq. (4) for the quark propagator with regard to the gauge identity (3) has been obtained in the present paper. These solutions are exact; however, the problem, as it has been formulated, limits their possible applicability range. Since we have used instead of the gluon propagator its infrared asymptotics, the solutions obtained can be applicable in the QCD infrared region only. It is important to emphasize the non-perturbative nature of our solutions. Besides, among them there are solutions both breaking the chiral invariance and conserving it. The analysis carried out by means of the effective potential method shows that, depending upon the values of the external parameters, different solutions may be preferable from the viewpoint of the minimal energy. As already mentioned, the presented solutions apparently do not cover the whole set of possible solutions. Therefore at this stage we do not attach a crucial importance to the arguments based on the comparison of the effective potentials. A more detailed physical interpretation of the solutions for the problem concerned requires further investigation.

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References

[1] S. Mandelstam, Phys. Rev. **D20** (1979) 3223.

[2] M. Baker, J.S. Ball and F. Zachariasen, Nucl. Phys. **B186** (1981) 531.

[3] H. Pagels, Phys. Rev. **D15** (1977) 2991.

[4] A.I. Alekseev, B.A. Arbuzov and V.A. Baykov, Teor. Mat. Fiz. **52** (1982) 187.

[5] A.I. Alekseev, B.A. Arbuzov and V.A. Baykov, Yad. Fiz. **34** (1981) 1374.

[6] K.R. Natroshvili, A.A. Khelashvili and V.Yu. Khmaladze, Teor. Mat. Fiz. **65** (1985) 360.

[7] B.A. Arbuzov, E.E. Boos and A.I. Davydychev, Teor. Mat. Fiz. **74** (1988) 163.

[8] B.A. Arbuzov and S.S. Kurennoy, Yad. Fiz. **36** (1982) 1314.

[9] B.A. Arbuzov, E.E. Boos, S.S. Kurennoy and K.Sh. Turashvili, Yad. Fiz. **42** (1985) 987.

[10] A.A. Slavnov, Teor. Mat. Fiz. **10** (1972) 153;
J.C. Taylor, Nucl. Phys. **B33** (1971) 436.

[11] E.S. Fradkin, Zh. Eksp. Teor. Fiz. **29** (1955) 258;
Y. Takahashi, Nuovo Cim. **6** (1957) 371.

[12] A.A. Slavnov, Teor. Mat. Fiz. **54** (1983) 52.

[13] A.V. Kulikov, M.L. Nekrasov and V.E. Rochev, Teor. Mat. Fiz. **65** (1985) 79.

[14] A.I. Alekseev, V.A. Baykov and E.E. Boos, Teor. Mat. Fiz. **54** (1983) 388.

[15] G. 't Hooft and M. Veltman, Nucl. Phys. **B44** (1972) 189;
C.G. Bollini and J.J. Giambiagi, Nuovo Cim. **B12** (1972) 20.

[16] B.A. Arbuzov, Dokl. Akad. Nauk SSSR **128** (1959) 1149.

[17] B.A. Arbuzov, preprint IHEP 85-55, Serpukhov, 1985.

[18] B.A. Arbuzov and A.T. Filippov, Nuovo Cim. **38** (1965) 796.

[19] J.M. Cornwall, R. Jackiw and E. Tomboulis, Phys. Rev. **D10** (1974) 2428.

[20] R. Casalbuoni, S. De Curtis, D. Dominici and R. Gatto, Phys. Lett. **B140** (1984) 357;
K. Stam, Phys. Lett. **B152** (1985) 238;
V.P. Gusynin and Yu.A. Sitenko, Z. Phys. **C29** (1985) 547.

[21] B.A. Arbuzov, E.E. Boos and K.Sh. Turashvili, Z. Phys. **C30** (1986) 287;
B.A. Arbuzov, E.E. Boos, S.S. Kurennoy and K.Sh. Turashvili, Yad. Fiz. **44** (1986) 1565.