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Transverse part of the quark-gluon vertex and quark propagator in infrared region

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Abstract

The Schwinger–Dyson equation for the quark propagator in QCD infrared region is investigated with due regard to gauge identity. The examples of possible transversal arbitrariness in quark-gluon vertex are presented. The account of the transverse part of the vertex is shown to be able to influence drastically the properties of the solutions obtained; in the general case an explicit expression for the solution is obtained, which depends on the form of the transverse part. Some particular cases and properties of corresponding solutions (non-perturbative character, chiral symmetry breaking, etc) are considered. A class of the solutions, which have massive poles with the mass m, is investigated by the effective potential method. It is shown that starting from some value of the quark bare mass m_0 the solutions with $m \neq 0$ become energy preferable.

Introduction

Much attention has recently been paid to the problems associated with the infrared behaviour of quantum chromodynamics. The inapplicability of perturbation theory in the infrared region complicates essentially the performed investigation and necessitates application of various non-perturbative methods. One of such methods is the approach developed in Refs. [1, 2, 3], which is based on studying the Schwinger–Dyson equations for appropriate Green's functions with due account of the Ward–Slavnov–Taylor gauge identities (see also review [4]). One of the results obtained in this way is the possibility of realizing the singular infrared gluon propagator asymptotics $D(k) \sim M^2/(k^2)^2$ (M is a mass dimension parameter). There are also some other independent arguments in favour of such a behaviour of the propagator (see, e.g., Ref. [4]).

Such an asymptotics in covariant gauge can be written as

$$D^{ab}_{\mu\nu}(k) \equiv \delta^{ab}\, D_{\mu\nu}(k) = \delta^{ab}\, \frac{M^2}{(k^2)^2} \left(g_{\mu\nu} - d\, \frac{k_\mu k_\nu}{k^2} \right) \tag{1}$$

where d is a gauge-fixing parameter and singularities in pseudo-Euclidean space are understood in the commonly adopted sense, namely, $1/(k^2)^\sigma \leftrightarrow 1/(k^2 + \mathrm{i}0)^\sigma$. It has been shown in Refs. [5, 6] that we can expect the simplest description of the Faddeev–Popov ghosts if we choose

$$d = \frac{4}{5-n}\,, \tag{2}$$

where n is a space-time dimension in the framework of dimensional regularization [7] (this d has first been used in paper [2], see also review [4]). In particular, the ghost contributions disappear in this case from the gauge identities for the gluonic Green's functions in the infrared limit [6]. These arguments allow us to assume that the ghost contributions to the gauge identity for the quark-gluon vertex are of no importance either in the infrared region in the considered case. It is also interesting that the Fourier transform of the propagator (1), (2) is transverse in the configuration space [5]. In Ref. [8] the two-loop terms of the Schwinger–Dyson equation for the gluon propagator, which are the most singular ones in the infrared region, have been considered, and the compatibility of the asymptotics (1), (2) with the equation has been proved. Note that similar studies in axial gauge have recently been carried out in Ref. [9].

Using the information about the gluon propagator in the infrared region, one can study the Schwinger–Dyson equation for the quark propagator. For example, in Refs. [5, 10, 11] some particular solutions to this equation have been obtained with expressions (1), (2) used as the gluon propagator. The quark propagator behaviour in some other cases (in particular, in other gauges) has been examined in many papers (see, e.g., [12, 13, 14]). The present paper continues and generalizes considerably the study of the quark propagator equation, which was started in Refs. [5, 10, 11]. The main subject of our paper is to obtain a sufficiently wide class of solutions to this equation and in association to study the influence of the transverse part of the quark-gluon vertex on the properties of the obtained solutions (sections 1 and 2). Section 3 shows also the possibility of applying the effective potential method to compare the solutions. The main results of the paper are briefly formulated in the conclusions.

1

Let us consider the Schwinger–Dyson equation for the quark propagator $G(p)$

$$1 = (\not p - m_0)\ G(p) + \frac{g^2 C_F}{(2\pi)^n \mathrm{i}} \int \mathrm{d}^n k\ D_{\mu\nu}(p-k)\ \gamma_\nu\ G(k)\ \Gamma_\mu(k,p;p-k)\ G(p)\ , \qquad (3)$$

where m_0 is the bare quark mass, C_F is a colour factor (which is equal to $(N_c^2-1)/(2N_c)$ for SU(N_c) group and to 4/3 for SU(3)), $n = 4+2\varepsilon$ is space-time dimension in the dimensional regularization framework. The quark-gluon vertex is defined as

$$\Gamma^a_\mu(p,q;k) = t^a\ \Gamma^a_\mu(p,q;k), \qquad\qquad k = q-p,$$

where t^a are the generators of the SU(N_c) group fundamental representation. We will use infrared asymptotics (1), (2) as gluon propagator $D_{\mu\nu}$.

Naturally, Eq. (3) itself is not sufficient to determine the Green's functions entering it, even if the information about the gluon propagator is taken from outside. Some additional information can be obtained from the Ward–Slavnov–Taylor gauge identity for the quark-gluon vertex, which, when ghost contributions are supposed to be of no importance in the infrared region, takes the well-known form of the Ward–Fradkin–Takahashi identity [15]

$$k_\mu\ \Gamma_\mu(p,q;k) = G^{-1}(q) - G^{-1}(p), \qquad k = q-p\ . \qquad (4)$$

This condition permits to reconstruct the vertex through the quark propagator except for the transverse (w.r.t. momentum k_μ) arbitrariness

$$\Gamma_\mu(p,q;k=q-p) = \Gamma^{(\mathrm{L})}_\mu + \Gamma^{(\mathrm{TR})}_\mu\ , \qquad (5)$$

where the longitudinal part $\Gamma^{(L)}_\mu$ satisfies identity (4), and

$$k_\mu\ \Gamma^{(\mathrm{TR})}_\mu(p,q;k) = 0, \qquad k = q-p\ . \qquad (6)$$

For a concrete definition of the longitudinal part of the vertex, it is convenient to use the representation which is equivalent to that obtained in Ref. [16],

$$\Gamma^{(\mathrm{L})}_\mu(p,q;q-p) = \frac{(\not p\gamma_\mu + \gamma_\mu \not q)G^{-1}(q) - G^{-1}(p)(\not p\gamma_\mu + \gamma_\mu \not q)}{q^2-p^2}\ . \qquad (7)$$

If, for example, the transverse part $\Gamma^{(\mathrm{TR})}_\mu$ was assumed to be of no importance in the infrared region (this assumption is repeatedly used in other papers), then after inserting ansatz (7) into Eq. (3) we should obtain a closed system of equations w.r.t. the functions entering the quark propagator. However, in this paper we will attempt to show that the account of the transverse part of the quark-gluon vertex can influence drastically the properties of the obtained solutions.

We first note that the transverse part $\Gamma^{(\mathrm{TR})}_\mu$ of the vertex can include the contributions $\Gamma^{(\mathrm{T})}_\mu$, for which the integral in (3) does not vanish (and which can influence in that way the properties of the obtained solutions), as well as the contributions which makes this integral vanish. The latter ones can be related to the arbitrariness in the vertex, $\Gamma_\mu^{(\mathrm{AR})}$:

$$\Gamma_\mu^{(\mathrm{TR})}(p,q;q-p) = \Gamma_\mu^{(\mathrm{T})} + \Gamma_\mu^{(\mathrm{AR})} \tag{8}$$

(such an arbitrariness can be added to all expressions obtained in the framework of the raised problem).

Let us present some examples of the arbitrariness $\Gamma_\mu^{(\mathrm{AR})}$. A functional arbitrariness of such a type was obtained earlier [10] (see also [11])

$$\Gamma_\mu^{(\mathrm{F})}(p,q;q-p) = F(p,q)\ G^{-1}(p)\ \left(\not{p}\gamma_\mu\not{q} - \not{q}\gamma_\mu\not{p}\right)\ G^{-1}(q)\ , \tag{9}$$

where $F(p,q)$ is an arbitrary symmetric scalar function. It can be also easily shown that there exists a functional arbitrariness of the form

$$\Gamma_\mu^{(\mathrm{f})}(p,q;k=q-p) = f(k^2)\ G^{-1}(p)\ \left(\gamma_\mu\not{k} - k_\mu\right)\ G^{-1}(q)\ , \tag{10}$$

(the expression in the brackets is proportional to $\sigma_{\mu\nu}k_\nu$). In addition, we succeeded in obtaining two examples of one-parametric arbitrariness:

$$\Gamma_\mu^{(\mathrm{a})}(p,q;k=q-p) = a\ \frac{G^{-1}(p)\ \not{p}}{p^2\sqrt{-p^2}}\left\{(\gamma_\mu(pk) - p_\mu\not{k})\not{q} - \not{p}\left(\gamma_\mu(qk) - q_\mu\not{k}\right)\right\}\frac{\not{q}\ G^{-1}(q)}{q^2\sqrt{-q^2}}\ , \tag{11}$$

$$\Gamma_\mu^{(\mathrm{b})}(p,q;k=q-p) = b\ G^{-1}(p)\ \left\{\frac{\not{p}\gamma_\mu\not{q}}{p^2q^2} + \left[\frac{(p+q)_\mu}{q^2-p^2} - \frac{2k_\mu}{k^2}\right]\left(\frac{\not{p}}{p^2} - \frac{\not{q}}{q^2}\right)\right\}\ G^{-1}(q), \tag{12}$$

where a and b are arbitrary constants. Expressions (9)–(12) are T-invariant.

It should be noted that the obtained expressions for the arbitrariness in the vertex have an essentially wider applicability range than the framework of the considered problem. First, these results can be generalized for the case of an arbitrary form of the gluon propagator, i.e.

$$D_{\mu\nu}(k) = D(k^2)g_{\mu\nu} + \{\text{term with tensor structure } k_\mu k_\nu\}. \tag{13}$$

One can note that formulae (9) and (10) can be retained without changes in this case, and expressions (11) and (12) should be multiplied by $[(k^2)^2\ D(k^2)]^{-1}$. Secondly, since transverse arbitrariness satisfies homogeneous identity (6), the ghost contributions are of no importance here. Therefore, the corresponding arbitrariness takes place, in particular, in QED.

Let us go over to the choice of $\Gamma_\mu^{(\mathrm{T})}$. We will use for it the representation containing the expression $(\not{p}\gamma_\mu\not{q} - \not{q}\gamma_\mu\not{p})$ (by analogy with [11]) sandwiched between the matrix of the following form:

$$\Gamma_\mu^{(\mathrm{T})}(p,q;q-p) = 2G^{-1}(p)\left[K(p^2)\not{p} + L(p^2)\right](\not{p}\gamma_\mu\not{q} - \not{q}\gamma_\mu\not{p})\left[K(q^2)\not{q} + L(q^2)\right]G^{-1}(q), \tag{14}$$

Due to the presence of arbitrary scalar functions K and L expression (14) has a sufficiently general form, although it does not cover all possibilities (a suitable example will be presented below).

Putting then expressions (7) and (14) into Eq. (3) and denoting the appropriate integrals as $I^{(\mathrm{L})}(p)$ and $I^{(\mathrm{T})}(p)$, we find

$$1 = (\not{p} - m_0)\left[A(p^2)\not{p} + B(p^2)\right] + \frac{g^2 C_F}{(2\pi)^4 \mathrm{i}}\Big(I^{(\mathrm{L})}(p) + I^{(\mathrm{T})}(p)\Big), \tag{15}$$

where the scalar functions at the matrix structures are picked out

$$G(p) = A(p^2)\not{p} + B(p^2). \tag{16}$$

To evaluate $I^{(\mathrm{L})}(p)$ we use (in analogy with Refs. [5, 11], see also [14]) the formula

$$\int \mathrm{d}^n k\, D_{\mu\nu}(p-k)\gamma_\nu\left[\not{k} f_1(k^2) + f_2(k^2)\right]\gamma_\mu = 6\mathrm{i}\pi^2 M^2\left[\not{p} f_1(p^2) - f_2(p^2)\right], \tag{17}$$

where the gluon propagator $D_{\mu\nu}$ is defined by Eqs. (1), (2), and the Fourier transform of the functions f_1 and f_2 can be defined as $n \to 4$; besides that a transition to the limit $n \to 4$ is performed in the right-hand side. Application of formula (17) yields

$$I^{(\mathrm{L})}(p) = 6\mathrm{i}\pi^2 M^2 A(p^2). \tag{18}$$

Using ansatz (14) and assuming the possibility of performing the limit $n \to 4$ (this assumption can be verified *a posteriori*), we easily obtain the expression

$$I^{(\mathrm{T})}(p) = 6\mathrm{i}\pi^2 M^2\left[K(p^2) + L(p^2)\not{p}/p^2\right]\, I_K(p^2), \tag{19}$$

where

$$I_K(p^2) = \frac{4}{3\mathrm{i}\pi^2}\int \frac{\mathrm{d}^4 k\, K(k^2)}{((p-k)^2)^2}\left(p^2k^2 - (pk)^2\right). \tag{20}$$

Note that the function L is absent in the integrand.

Passing to the Euclidean variables ($x \equiv p_{\mathrm{E}}^2 = -p^2$, $y \equiv k_{\mathrm{E}}^2 = -k^2$) and performing the angular integration (see, e.g., Refs. [11, 17]), one can find

$$I_K(x) = \frac{1}{x}\int_0^x y^2 \mathrm{d}y\, K(y) + x\int_x^\infty \mathrm{d}y\, K(y). \tag{21}$$

Then, taking into account (18), (19) and equating the expressions afore the corresponding matrix structures in Eq. (15), we get the simultaneous equations

$$\left.\begin{aligned} 1 &= (p^2 + \kappa^2)A(p^2) - m_0 B(p^2) + \kappa^2 K(p^2) I_K(p^2) \\ 0 &= B(p^2) - m_0 A(p^2) + \frac{\kappa^2}{p^2} L(p^2) I_K(p^2) \end{aligned}\right\} \tag{22}$$

where the parameter κ is defined as

$$\kappa^2 = \frac{3g^2M^2C_F}{8\pi^2} \qquad \left(\text{for } \mathrm{SU}_c(3),\quad \kappa^2 = \frac{g^2M^2}{2\pi^2}\right). \tag{23}$$

This parameter can be connected with the experimentally defined slope of the linear part of the quark-antiquark potential in potential models. The usual value is $\kappa \simeq 0.42$ GeV [2, 4].

In particular, we can obtain from system (22) the expressions for $A(p^2)$ and $B(p^2)$ (see (16)) in terms of the functions entering $\Gamma_\mu^{(\mathrm{T})}$ (14):

$$A(p^2) = \frac{1}{p^2 - m_0^2 + \kappa^2}\left\{1 - \kappa^2\left[K(p^2) + \frac{m_0}{p^2}L(p^2)\right] I_K(p^2)\right\}, \tag{24}$$

$$B(p^2) = \frac{1}{p^2 - m_0^2 + \kappa^2}\left\{m_0 - \kappa^2\left[m_0 K(p^2) + \frac{p^2+\kappa^2}{p^2}L(p^2)\right] I_K(p^2)\right\}. \tag{25}$$

If, for example, $K(p^2) = 0$, then we come to the solution obtained in Ref. [10], i.e.

$$G(p) = \frac{\not{p} + m_0}{p^2 - m_0^2 + \kappa^2}. \tag{26}$$

The corresponding $\Gamma_\mu^{(\mathrm{T})}$ (see (14)) then can be related to the functional arbitrariness $\Gamma_\mu^{(\mathrm{F})}$ (9), and, therefore, one can consider $\Gamma_\mu^{(\mathrm{T})} = 0$.

The analysis of a great number of possibilities shows that a simple form of the solution, which satisfies, firstly, the boundary conditions for integral Eqs. (22) (see [11]) and, secondly, the applicability conditions of formula (17), can be obtained in the case when

$$K(x) = cx^{-3/2}, \qquad c = \text{const}.$$

Expressions (23), (24) take then the form

$$A(p^2) = \frac{1}{p^2 - m_0^2 + \kappa^2}\left\{1 + \frac{\kappa^2\lambda}{p^2} - \frac{m_0\kappa^2}{p^2}N(p^2)\right\}, \tag{27}$$

$$B(p^2) = \frac{1}{p^2 - m_0^2 + \kappa^2}\left\{m_0 + \frac{m_0\kappa^2\lambda}{p^2} - \frac{\kappa^2(p^2+\kappa^2)}{p^2}N(p^2)\right\}, \tag{28}$$

where the following notations are introduced:

$$N(p^2) = L(p^2) I_K(p^2), \qquad \lambda = 8c^2/3. \tag{29}$$

It should be noted that $N(p^2)$ is in fact an arbitrary function, since the integral equations do not impose any conditions on $L(p^2)$ (one must check only the fulfilment of the applicability conditions of formula (17); in particular, this requirement excludes multifold poles since Fourier transforms of the relevant functions cannot be defined as $n \to 4$).

It is interesting to note that in the chiral limit, $m_0 \to 0$, we find from (27) and (28)

$$A(p^2) = \frac{1}{p^2 + \kappa^2}\left\{1 + \frac{\kappa^2\lambda}{p^2}\right\}, \qquad B(p^2) = -\frac{\kappa^2}{p^2}N(p^2), \tag{30}$$

i.e. the function N corresponds to the chiral-symmetry-breaking part of the quark propagator. A non-physical "tachyonic" pole in $A(p^2)$ can be removed by putting $\lambda = 1$. Then the expression for the quark propagator has the form

$$G(p) = \frac{\not{p} - \kappa^2 N(p^2)}{p^2} \qquad (m_0 = 0), \tag{31}$$

which differs from the free propagator only by the chiral-symmetry-breaking term with an arbitrary (with the mentioned reserve) function. In the case when $N(p^2)$ decreases sufficiently quickly at high Euclidean momenta, expression (31) can simulate the transition of the quark propagator to its ultraviolet behaviour. It is interesting that, choosing $N(p^2)$ in a suitable form, one can get here a particular case of the expression for the quark propagator, which has been obtained in Ref. [14] with a different prescription for the gluon propagator singularity.

2

Let us investigate now some particular cases of expressions (27), (28). Consider first the case $N(p^2) = \nu = \text{const}$. We find then

$$\left.\begin{aligned} G(p) &= (1-\alpha)\frac{\not p + m_0}{p^2 - m_0^2 + \kappa^2} + \frac{\alpha\not p + \gamma\kappa}{p^2}, \\ \Gamma_\mu^{(\mathrm{T})}(p,q;q-p) &= \frac{3}{4\kappa^3(\alpha\kappa - \gamma m_0)}\frac{G^{-1}(p)\not p}{p^2\sqrt{-p^2}}\left[(\gamma\kappa - \alpha m_0)\not p + \kappa(\alpha\kappa - \gamma m_0)\right] \\ &\quad\times\left(\not p\gamma_\mu\not q - \not q\gamma_\mu\not p\right)\left[(\gamma\kappa - \alpha m_0)\not q + \kappa(\alpha\kappa - \gamma m_0)\right]\frac{\not q G^{-1}(q)}{q^2\sqrt{-q^2}} \end{aligned}\right\} \tag{32}$$

where the following notations are used:

$$\alpha = \frac{\kappa^2(m_0\nu - \lambda)}{m_0^2 - \kappa^2}, \qquad \gamma = \frac{\kappa^2(\kappa^2\nu - m_0\lambda)}{m_0^2 - \kappa^2}$$

Propagator (32) includes two terms, one of them is of form (26) and contains a massive pole (the squared mass has the "right" sign at $m_0^2 \geq k^2$), and the second one has a zero-mass pole. It can easily be seen that, for special values of parameters α and γ, we can get from (32) all particular solutions, which have been obtained earlier in papers [5, 10, 11].

Expression (32) for $\Gamma_\mu^{(\mathrm{T})}$ shows that the case $\alpha\kappa = \gamma m_0$, which corresponds to the quark propagator

$$G(p) = (1-\alpha)\frac{\not p + m_0}{p^2 - m_0^2 + \kappa^2} + \alpha\frac{\not p + \kappa^2/m_0}{p^2} \tag{33}$$

requires a special examination. Let us carry the out transition to the limit $\gamma m_0 \to \alpha\kappa$ in $\Gamma_\mu^{(\mathrm{T})}$. For this purpose we multiply the terms in square brackets. Then the obtained term of the order of $(\alpha\kappa - \gamma m_0)$ tends to zero, and it can easily be seen that the singular term of the order of $(\alpha\kappa - \gamma m_0)^{-1}$ corresponds to the functional arbitrariness $\Gamma_\mu^{(\mathrm{F})}$ (9) and it can be subtracted from the general expression. Therefore, we obtain for $\gamma m_0 = \alpha\kappa$

$$\Gamma_\mu^{(\mathrm{T})}(p,q;q-p) = \frac{3\alpha(\kappa^2 - m_0^2)}{4m_0\kappa^2}\frac{G^{-1}(p)\not p}{p^2\sqrt{-p^2}}\left[\not p\left(\not p\gamma_\mu\not q - \not q\gamma_\mu\not p\right) + \left(\not p\gamma_\mu\not q - \not q\gamma_\mu\not p\right)\not q\right]\frac{\not q G^{-1}(q)}{q^2\sqrt{-q^2}} \tag{34}$$

By analogous subtraction of the term corresponding to $\Gamma_\mu^{(\mathrm{F})}$ (9), we can obtain from (32) a general expression which is correct for any α and γ,

$$\begin{aligned} \Gamma_\mu^{(\mathrm{T})}(p,q;q-p) &= \frac{3(\gamma\kappa - \alpha m_0)}{4\kappa^2}\frac{G^{-1}(p)\not p}{p^2\sqrt{-p^2}}\left[\not p\left(\not p\gamma_\mu\not q - \not q\gamma_\mu\not p\right) + \left(\not p\gamma_\mu\not q - \not q\gamma_\mu\not p\right)\not q\right]\frac{\not q G^{-1}(q)}{q^2\sqrt{-q^2}} \\ &\quad + \frac{3(\alpha\kappa - \gamma m_0)}{4\kappa}\frac{G^{-1}(p)\not p}{p^2\sqrt{-p^2}}\left(\not p\gamma_\mu\not q - \not q\gamma_\mu\not p\right)\frac{\not q G^{-1}(q)}{q^2\sqrt{-q^2}}. \end{aligned} \tag{35}$$

One should emphasize a non-perturbative character of expressions (31) and (35), which contain $\kappa \sim g$ in denominators.

It should be also noted that propagator (32) can be obtained with a different choice of the transverse part of the quark-gluon vertex, namely

$$\Gamma_\mu^{(\mathrm{T})}(p,q;k=q-p) = 2G^{-1}(p)\left[\tilde{K}(p^2)\not{p} + \tilde{L}(p^2)\right](\gamma_\mu \not{k} - k_\mu)\left[\tilde{K}(q^2)\not{q} + \tilde{L}(q^2)\right]G^{-1}(q), \tag{36}$$

which differs from (14) in the form of the structure placed between the expressions in square brackets ($(\gamma_\mu \not{k} - k_\mu) \sim \sigma_{\mu\nu}k_\nu$). After similar consideration we get convinced that a solution of simple form takes place when

$$\tilde{K}(x) = 0, \qquad \tilde{L}(x) = \tilde{c}x^{-1/2}, \qquad \tilde{c} = \mathrm{const}.$$

The corresponding propagator coincides with (32), and

$$\Gamma_\mu^{(\mathrm{T})}(p,q;q-p) = \frac{3(\kappa^2 - m_0^2)\alpha}{m_0\kappa^2}\,\frac{G^{-1}(p)}{\sqrt{-p^2}}\,(\gamma_\mu \not{k} - k_\mu)\,\frac{G^{-1}(q)}{\sqrt{-q^2}}. \tag{37}$$

where the parameter α is connected with $\tilde{c}$ in the following way:

$$\alpha = \frac{2m_0\kappa^2\tilde{c}^2}{3(\kappa^2 - m_0^2)}.$$

It can easily be seen that $\Gamma_\mu^{(\mathrm{T})}$ (37) differs from $\Gamma_\mu^{(\mathrm{T})}$ (34) in the structure of the form $\Gamma_\mu^{(a)}$ (11), which corresponds to the arbitrariness in Eq. (3).

The pole with an arbitrary mass m can also be obtained in expressions (26), (27) for the quark propagator. For this purpose we choose the function $N(p^2)$ in the form

$$N(p^2) = \frac{\mu}{p^2 - m^2} + \nu, \qquad (\mu, \nu = \mathrm{const}).$$

Then we get

$$\left.\begin{aligned}
G(p) &= (1-\alpha-\beta)\,\frac{\not{p} + m_0}{p^2 - m_0^2 + \kappa^2} + \frac{\alpha\not{p} + \gamma\kappa}{p^2} + \beta\frac{\not{p} + m'}{p^2 - m^2}\,, \\
\Gamma_\mu^{(\mathrm{T})}(p,q;q-p) &= \frac{3}{4\kappa^3(\alpha\kappa - m_0\gamma)}\,\frac{G^{-1}(p)}{\sqrt{-p^2}} \\
&\times\left[\kappa(\alpha\kappa - m_0\gamma)\frac{\not{p}}{p^2} + \gamma\kappa - \alpha m_0 + \beta(m' - m_0)\frac{p^2}{p^2 - m^2}\right](\not{p}\gamma_\mu\not{q} - \not{q}\gamma_\mu\not{p}) \\
&\times\left[\kappa(\alpha\kappa - m_0\gamma)\frac{\not{q}}{q^2} + \gamma\kappa - \alpha m_0 + \beta(m' - m_0)\frac{q^2}{q^2 - m^2}\right]\frac{G^{-1}(q)}{\sqrt{-q^2}}\,,
\end{aligned}\right\} \tag{38}$$

where $m' \equiv (m^2 + \kappa^2)/m_0$, and the parameters α, β, γ are defined by

$$\alpha = \frac{\kappa^2(m_0\nu - \lambda - m_0\mu/m^2)}{m_0^2 - \kappa^2}, \quad \beta = -\frac{\kappa^2 m_0\mu}{m^2(m^2 - m_0^2 + \kappa^2)}, \quad \gamma = \frac{\kappa(\kappa^2\nu - \kappa^2\mu/m^2 - \lambda m_0)}{m_0^2 - \kappa^2}.$$

A "special case" $\kappa\alpha = m_0\gamma$ can be removed again just in the same way, by omitting the part corresponding to $\Gamma_\mu^{(\mathrm{F})}$. An appropriate expression for $\Gamma_\mu^{(\mathrm{T})}$, which is correct for any values of α, β and γ, is defined by

$$\begin{aligned}\Gamma_\mu^{(\mathrm{T})}(p,q;q-p) &= \frac{3}{4\kappa^2}\frac{G^{-1}(p)}{\sqrt{-p^2}}\left\{\frac{\not{p}}{p^2}\left(\not{p}\gamma_\mu\not{q}-\not{q}\gamma_\mu\not{p}\right)\left[\gamma\kappa-\alpha m_0+\beta(m'-m_0)\frac{q^2}{q^2-m^2}\right]\right.\\ &\left.+\left[\gamma\kappa-\alpha m_0+\beta(m'-m_0)\frac{p^2}{p^2-m^2}\right]\left(\not{p}\gamma_\mu\not{q}-\not{q}\gamma_\mu\not{p}\right)\frac{\not{q}}{q^2}\right\}\frac{G^{-1}(q)}{\sqrt{-q^2}}\\ &+\frac{3(\alpha\kappa-m_0\gamma)}{4\kappa}\frac{G^{-1}(p)\not{p}}{p^2\sqrt{-p^2}}\left(\not{p}\gamma_\mu\not{q}-\not{q}\gamma_\mu\not{p}\right)\frac{\not{q}G^{-1}(q)}{q^2\sqrt{-q^2}}\,.\end{aligned} \tag{39}$$

Expressions (38), (39) are rather cumbersome and have a lot of free parameters (α,β,γ,m). The choice of the special values of these parameters will leave only the pole on an arbitrary mass m (we must set $\alpha=\gamma=0$, $\beta=1$ for that). In this case expression (39) should be used for $\Gamma_\mu^{(\mathrm{T})}$, because formula (38) has a singularity at $\alpha=\gamma=0$. As a result, using Eq. (7) for $\Gamma_\mu^{(\mathrm{L})}$ we find

$$\left.\begin{aligned}&G(p)=\frac{\not{p}+(m^2+\kappa^2)/m_0}{p^2-m^2}=\frac{p+m'}{p^2-m^2},\\ &\Gamma_\mu^{(\mathrm{L})}(p,q;q-p)=\gamma_\mu+(m^2-m'^2)(\not{p}+m')^{-1}\gamma_\mu(\not{q}+m')^{-1},\\ &\Gamma_\mu^{(\mathrm{T})}(p,q;q-p)=\frac{3(m'-m_0)}{4\kappa^2}\frac{G^{-1}(p)}{\sqrt{-p^2}}\left[\frac{\not{p}}{p^2}\left(\not{p}\gamma_\mu\not{q}-\not{q}\gamma_\mu\not{p}\right)\frac{q^2}{q^2-m^2}\right.\\ &\qquad\left.+\frac{p^2}{p^2-m^2}\left(\not{p}\gamma_\mu\not{q}-\not{q}\gamma_\mu\not{p}\right)\frac{\not{q}}{q^2}\right]\frac{G^{-1}(q)}{\sqrt{-q^2}}.\end{aligned}\right\} \tag{40}$$

For $m'=m_0$ $(m^2=m_0^2\kappa^2)$ $\Gamma_\mu^{(\mathrm{T})}=0$ and we come to solution (26), and for $m=0$ the solution of form (32) with $\alpha=1$ is obtained. Note that for $p^2=q^2=m^2$ $\Gamma_\mu^{(\mathrm{T})}$ (40) vanishes since the inverse propagator gives the factor $(p^2-m^2)(q^2-m^2)$. In addition, it is seen that the massive solutions of Dirac form can be obtained here, if the condition $m'\equiv(m^2+\kappa^2)/m_0=m$ is satisfied.

Note, that for massive solutions of this type the term, contributing to the quark chromomagnetic moment [18] can be obtained from arbitrariness (9) ($F(p,q)\sim(p^2-m^2)^{-1}(q^2-m^2)^{-1}$ should be chosen for that), since the result of the action the structure $(\not{p}\gamma_\mu\not{q}-\not{q}\gamma_\mu\not{p})$ on spinor states is equivalent to that for the chromomagnetic moment structure $\sigma_{\mu\nu}k_\nu$ $(k=q-p)$. The coefficient of the expression, which yields the chromomagnetic moment value, remains arbitrary in this case, i.e. it cannot be extracted from the propagator equation. Apparently, the equation for the quark-gluon vertex should be used to find the chromomagnetic moment (an investigation of this kind was carried out in Ref. [18]).

3

When investigating the Green's functions, containing free parameters or functions, the method of effective potential for composite operators [19] proves to be convenient to find preferable solutions. The effective potential for the propagator $G(p)$, which is the solution to the relevant Schwinger–Dyson equation, can be defined by expression [20] (see also [11])

$$V_{\rm eff}[G] = \frac{1}{2{\rm i}(2\pi)^4}\int {\rm d}^4p\ {\rm Tr}\left[2\ln(G_0^{-1}G) - G_0^{-1}G + 1\right], \qquad (41)$$

where $G_0(p) = (p - m_0)^{-1}$ is the free quark propagator. Since the obtained solutions can be correct in the infrared region only, we (in analogy with paper [11]) will restrict the integration domain in (41) by the condition $p_{\rm E}^2 < \Lambda^2$, where Λ is an effective cut-off confining the infrared region. One can expect the value of Λ correspond to the value of the characteristic momentum k_0 separating the infrared region from the ultraviolet one. The definition of the momentum k_0 has been discussed, in particular, in Ref. [21], where its value has been estimated as $k_0 \sim 0.7$ GeV. Following these arguments we will consider $\Lambda \sim k_0$.

Let us consider, as an example, the application of the effective potential method to solution (40). The insertion of propagator (40) into (41) yields

$$\begin{aligned} V_{\rm eff}(z,z_0) \ &= \ \frac{N_c\kappa^4}{8\pi^2}\Bigg\{\frac{w^2}{2}\ln\left[\frac{(w+z_0)\,(w+(z+1)^2/z_0)}{(w+z)^2}\right] - \frac{z_0^2}{2}\ln\left[\frac{w+z_0}{z_0}\right] \\ &-\frac{(z+1)^4}{2z_0^2}\ln\left[\frac{w+(z+1)^2/z_0}{(z+1)^2/z_0}\right] + z(z+1)\ln\left[\frac{w+z}{z}\right] + \frac{w(z+1-z_0)^2}{2z_0}\Bigg\}, \end{aligned} \qquad (42)$$

where dimensionless variables $z = m^2/\kappa^2$, $z_0 = m_0^2/\kappa^2$ and $w = \Lambda^2/\kappa^2$ are introduced (dimensionless-making parameter κ is defined by formula (23)). Taking into account the cited arguments about κ and Λ ($\kappa \simeq 0.42$ GeV, $\Lambda \simeq 0.7$ GeV) we have $w \simeq 2.78$. Note that the variation of w within a sufficiently wide range does not lead to qualitative changes in the picture. Since m_0 is an external parameter in our problem, we will look for the minimum of $V_{\rm eff}(z,z_0)$ with respect to z for fixed values of z_0, i.e. we will vary, in fact, w.r.t. the mass m.

The numerical study of the effective potential (42) leads to the following results. When $0 < z_0 < 1.33$, $V_{\rm eff}(z,z_0)$ as a function of z is minimal for $z = 0$. Starting from $z_0 \simeq 1.16$, however, one additional minimum of this function appears at the point $z_m(z_0) \neq 0$ (see Fig. 1). When $z_0 > 1.33$, this local minimum becomes the global one, i.e. $V_{\rm eff}(z_m(z_0),z_0) < V_{\rm eff}(0,z_0)$ (see Fig. 2). So, for the value of the external parameter $z_0 \simeq 1.33$ ($m_0 = m_0^{\rm cr} \simeq 0.48$ GeV) there occurs a phase transition from the solution with a singularity at $p^2 = 0$ to the massive solution, which becomes energy preferable. It is interesting that the dynamic mass value of $m \simeq 0.34$ GeV, which agrees well with commonly used values for constituent masses of light quarks, corresponds to the phase transition point.

The possibility, and even desirability, of the phase transition from the current quark mass to the constituent one has been repeatedly mentioned in papers (see, for example, [22]). It has been pointed out that such a transition at $Q^2 \sim 1$ GeV^2 improves the description of quarkonia spectra in potential models [23].

Discussing the connection of the obtained results with the phenomenology, one must take into account that in the framework of our approach studying the infrared region, we include into m_0 contributions from the ultraviolet region. Thus, m_0 becomes a running

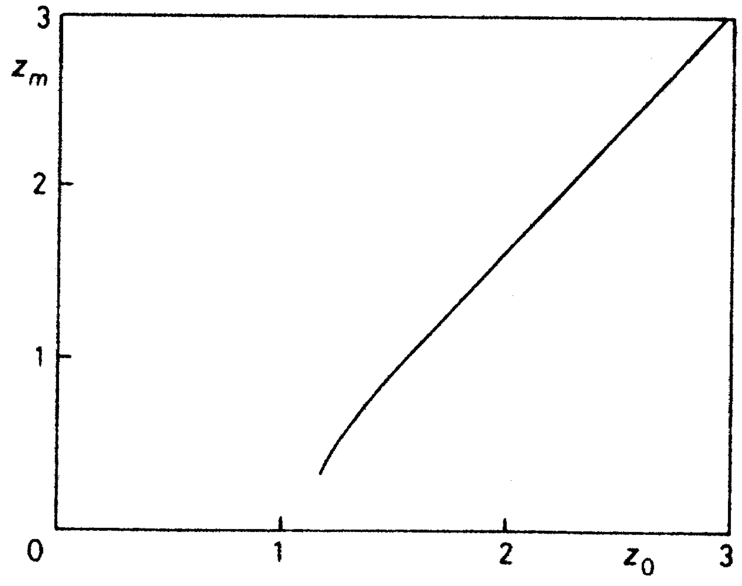

Figure 1: Position of the local minimum of effective potential (z_m) versus z_0.

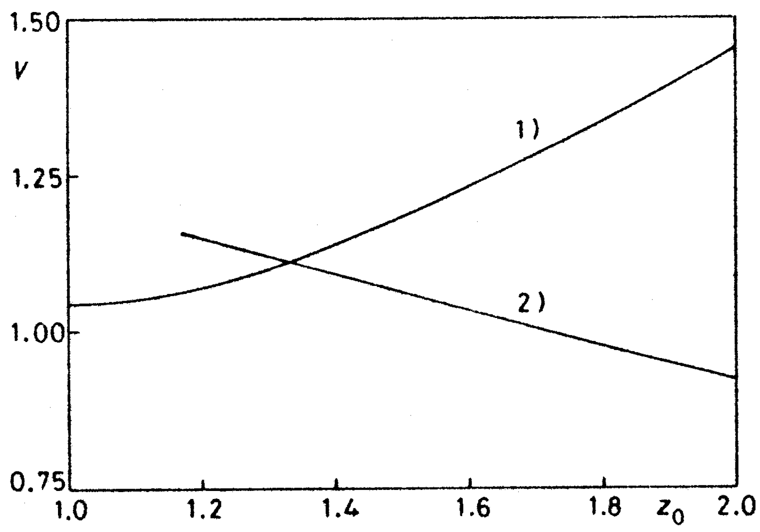

Figure 2: Effective potential values (divided by $N_c\kappa^4/(8\pi^2)$) in minimum points 1) $z = 0$ and 2) $z = z_m$ vs. z_0.

mass, $m_0(Q^2)$, where Q^2 is a normalization point. The running mass is known to increase slowly when Q^2 decreases [24], and the situation may be realized when for some Q^2 the mass $m_0(Q^2)$ takes through the value of $m_0^{\rm cr}$. Then the obtained above phase transition from the massless solution to the massive one takes place at this point. If $Q_0^2 \sim 1$ GeV^2 we have a qualitative description of the phenomenological picture.

It should also be noted that the examined solution (40) gives a possibility to estimate the value of quark condensate $\langle \bar{q}q \rangle$. Indeed, when calculating the condensate, the values of parameters entering it are usually taken at the normalization point $Q^2 \sim 1$ GeV^2, i.e. near the supposed phase transition point, where $m_0 = m_0^{\rm cr}$. Considering $Q^2 \geq Q_0^2$ which corresponds to the case of current (small) quark mass, we take in (40) $m = 0$, $m_0 = m_0^{\rm cr}$ and obtain the estimate

$$\langle \bar{q}q \rangle = \frac{N_c}{4\pi^2} \int\limits_0^{\Lambda^2} {\rm d}y \; y \; B(y) = -\frac{N_c \kappa^2 \Lambda^2}{4\pi^2 m_0^{\rm cr}} \simeq -(0.24 \text{ GeV})^3 \;, \tag{43}$$

which is an a good agreement with the phenomenology (see, e.g., [24]). It is interesting that with analogous calculation of $\langle \bar{q}q \rangle$ with massive solution (40) for $Q^2 \leq Q_0^2$, taking the values of parameters m_0 and m near the phase transition point ($m_0 = m_0^{\rm cr}$, $m \simeq 0.34$ GeV), we get the same values as in (43), i.e. the condensate value does not change at the considered phase transition.

Conclusions

The present paper is devoted to the investigation of the Schwinger–Dyson equation for the quark propagator in the infrared region. The examples of a possible transverse arbitrariness have been presented. The expressions for this arbitrariness can be generalized for the case of an arbitrary form of gluon or, in QED, photon propagator. A sufficiently general form of representation for the transverse part (14) of the quark-gluon vertex, which contains two arbitrary functions, has been considered, and the general form of solution (24), (25) has been obtained in this case. This solution includes the integral operator (20) and takes the simplest form for a special choice of the function $K(p^2)$ entering the integrand. Appropriate expressions (27), (28) contain the function $N(p^2)$ (29), which is connected with the ansatz for the transverse part of the quark-gluon vertex. At the chiral limit (as $m_0 \to 0$) this function corresponds to the chiral-symmetry-breaking structure (see (31)). Thus, the performed study shows that the account of the transverse part of the quark-gluon vertex can influence drastically the properties of the solutions obtained for the quark propagator in the infrared region. An explicit form of the solutions, which depends on the functions entering the transverse part (14), is presented in Eqs. (24), (25) and (27), (28).

Then, some particular choices of the function $N(p^2)$ and the properties of the relevant solutions (non-perturbative character, etc.) have been considered. In particular, the possibility of the solutions having poles with the mass m (38) to exist, has been shown. The explicit form of such solution is presented by Eq. (40). The application of the effective potential method to solution (40) has led to the interesting conclusion that, beginning with some value of the bare quark mass m_0, the solutions with $m \neq 0$ become energy preferable. It gives a possibility to describe the phase transition from the current quark mass to the constituent one. A good agreement of the value for quark mass "jump", $m \simeq 0.34$ GeV, and of the quark condensate value (43) with the phenomenology can be considered as an additional argument in favour of the fact that the obtained solutions to a certain extent correspond to the real situation.

It should be noted, however, that the arbitrariness in choosing the transverse part of the vertex does not give a possibility of more concrete physical interpretation of these solutions at the stage. Perhaps, a study of equations for vertex functions would allow us to reduce this arbitrariness.

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