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## **A method for calculating vertex-type Feynman integrals**

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### **Abstract**

A method is proposed for exact calculation of the dimensionally regulated vertex-type Feynman diagrams. This method is used to obtain expressions for a class of massless vertex integrals which extend the number of exactly calculable Feynman diagrams in the quantum field theory and which are of interest for calculations in gauge theories and in studying processes of elementary particles interaction at high energies.

1. For obtaining various physical results in the quantum field theory, by using either perturbative or non-perturbative methods (in particular, when investigating the infrared region of quantum chromodynamics), it is very important to know how to exactly calculate the widest possible class of Feynman diagrams. Calculation of massless Feynman integrals is of special interest since, first, one has to deal with massless particles (photons, gluons, ghosts, etc.) in gauge theories and, second, when studying high-energy processes the masses of the particles can often be neglected as compared to their momenta.

When calculating Feynman diagrams in gauge theories, it is most convenient to employ the method of dimensional regularization [1] which, in particular, makes it possible to preserve the gauge invariance at all stages of the calculation. In some papers (see, for instance, [2, 3]) expressions have been obtained for massless propagator-type (i.e., depending on a single external momentum) integrals. The present paper is devoted to the calculation, in the framework of the dimensional regularization, of one-loop massless Feynman integrals belonging to the vertex type (i.e., depending on two external momenta).

2. The following type of integrals is examined:

$$J(\mu, \nu, \rho) = \int \frac{d^n r}{(r^2 + i0)^\mu ((p-r)^2 + i0)^\nu ((q-r)^2 + i0)^\rho}, \quad (1)$$

where  $n = 4 + 2\varepsilon$  is the space-time dimension,  $\mu$ ,  $\nu$  and  $\rho$  are arbitrary parameters, whereas the infinitesimal imaginary additions in the denominators determine the rules of passing the poles in the pseudo-Euclidean space.

Using the  $\alpha$ -representation [4] for the denominators in Eq. (1), integrating over the momentum  $r$  and performing a standard substitution of the integration variables, we find the following expression:

$$\begin{aligned} J(\mu, \nu, \rho) &= \pi^{n/2} \frac{i^{1-\mu-\nu-\rho-n/2}}{\Gamma(\mu) \Gamma(\nu) \Gamma(\rho)} \int_0^\infty d\lambda \lambda^{\mu+\nu+\rho-n/2-1} \\ &\times \int_0^1 d\xi \xi^{\nu-1} (1-\xi)^{\mu-1} \int_0^1 d\eta \eta^{\rho-1} (1-\eta)^{\mu+\nu-1} \\ &\times \exp \left[ i\lambda(1-\eta)(\xi(1-\xi)(1-\eta)p^2 + (1-\xi)\eta q^2 + \xi\eta k^2) \right], \quad (2) \end{aligned}$$

where  $k = q - p$ . Using the known formulae [5], the  $\lambda$ - and  $\eta$ -integrals can be calculated. As a result, we obtain

$$\begin{aligned} J(\mu, \nu, \rho) &= \pi^{n/2} i^{1-n} \frac{\Gamma(\mu + \nu + \rho - n/2) \Gamma(n/2 - \rho)}{\Gamma(\mu) \Gamma(\nu) \Gamma(n/2)} (p^2)^{n/2-\mu-\nu-\rho} \\ &\times \int_0^1 d\xi \xi^{n/2-\mu-\rho-1} (1-\xi)^{n/2-\nu-\rho-1} \\ &\times {}_2F_1 \left( \begin{matrix} \rho, \mu + \nu + \rho - n/2 \\ n/2 \end{matrix} \middle| 1 - \frac{(1-\xi)q^2 + \xi k^2}{\xi(1-\xi)p^2} \right), \quad (3) \end{aligned}$$

where  ${}_2F_1$  is the Gauss hypergeometric function. To calculate the remaining  $\xi$ -integral in Eq. (3), let us use the formulae of analytic continuation of the hypergeometric functions [6],

and then represent them as Mellin–Barnes integrals,

$${}_2F_1 \left( \begin{matrix} a, b \\ c \end{matrix} \middle| z \right) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} ds (-z)^s \frac{\Gamma(-s)\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)},$$

where the integration contour is chosen so as to separate “right” and “left” poles of the occurring  $\Gamma$  functions. (In what follows, all contour integrals will be understood in this sense.) Now, the  $\xi$ -integrals can be calculated [5] and, again applying the analytic continuation formulae, we obtain an expression consisting of four terms. For instance, the first term looks like

$$\begin{aligned} & \pi^{n/2} i^{1-n} \frac{\Gamma(n-\mu-\nu-2\rho)\Gamma(\mu+\nu+2\rho-n+1)}{\Gamma(\mu)\Gamma(\nu)\Gamma(\rho)\Gamma(n-\mu-\nu-\rho)} \left\{ \Gamma(n/2-\nu-\rho) (p^2)^{n/2-\mu-\nu-\rho} \right. \\ & \times \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} ds \left( -\frac{q^2}{p^2} \right)^s \frac{\Gamma(-s)\Gamma(n/2-\mu-\rho-s)\Gamma(\rho+s)\Gamma(\mu+\nu+\rho-n/2+s)}{\Gamma(n-\mu-\nu-2\rho-s)\Gamma(\mu+\nu+2\rho-n+1+s)} \\ & \left. \times {}_2F_1 \left( \begin{matrix} -s, n/2-\mu-\rho-s \\ \nu+\rho-n/2+1 \end{matrix} \middle| \frac{k^2}{q^2} \right) \right\}. \end{aligned} \quad (4)$$

Using the fact that

$$\frac{1}{\Gamma(c)} {}_2F_1 \left( \begin{matrix} a, b \\ c \end{matrix} \middle| z \right)$$

is an analytical function of its parameters [6], one can express the contour integrals in Eq. (4) as the sums over the poles of the  $\Gamma$  functions from the integrand. Closing the integration contour to the right, we obtain a sum of six such series. The obtained expression can be essentially simplified, by means of known formulae for the  $\Gamma$  function [6]. As a result of these transformations, we obtain a sum of four series,

$$\begin{aligned} J(\mu, \nu, \rho) &= \pi^{n/2} i^{1-n} \frac{\Gamma(n/2-\mu-\rho)\Gamma(\mu+\rho-n/2+1)}{\Gamma(\mu)\Gamma(\nu)\Gamma(\rho)\Gamma(n-\mu-\nu-\rho)} \\ & \times \left\{ (p^2)^{-\mu} (k^2)^{n/2-\nu-\rho} \Gamma(\nu+\rho-n/2) \right. \\ & \times \sum_{j=0}^{\infty} \frac{1}{j!} \left( \frac{q^2}{p^2} \right)^j \frac{\Gamma(\mu+j)\Gamma(n/2-\nu+j)}{\Gamma(\mu+\rho-n/2+1+j)} {}_2F_1 \left( \begin{matrix} -j, n/2-\mu-\rho-j \\ n/2-\nu-\rho+1 \end{matrix} \middle| \frac{k^2}{q^2} \right) \\ & - (p^2)^{-\nu} (q^2)^{n/2-\mu-\rho} \Gamma(n/2-\nu-\rho) \\ & \times \sum_{j=0}^{\infty} \frac{1}{j!} \left( \frac{q^2}{p^2} \right)^j \frac{\Gamma(\nu+j)\Gamma(n/2-\mu+j)}{\Gamma(n/2-\mu-\rho+1+j)} {}_2F_1 \left( \begin{matrix} -j, \mu+\rho-n/2-j \\ \nu+\rho-n/2+1 \end{matrix} \middle| \frac{k^2}{q^2} \right) \\ & + (p^2)^{n/2-\mu-\nu-\rho} \Gamma(n/2-\nu-\rho) \\ & \times \sum_{j=0}^{\infty} \frac{1}{j!} \left( \frac{q^2}{p^2} \right)^j \frac{\Gamma(\rho+j)\Gamma(\mu+\nu+\rho-n/2+j)}{\Gamma(\mu+\rho-n/2+1+j)} {}_2F_1 \left( \begin{matrix} -j, n/2-\mu-\rho-j \\ \nu+\rho-n/2+1 \end{matrix} \middle| \frac{k^2}{q^2} \right) \\ & - (p^2)^{\rho-n/2} (q^2)^{n/2-\mu-\rho} (k^2)^{n/2-\nu-\rho} \Gamma(\nu+\rho-n/2) \\ & \left. \times \sum_{j=0}^{\infty} \frac{1}{j!} \left( \frac{q^2}{p^2} \right)^j \frac{\Gamma(n-\mu-\nu-\rho+j)\Gamma(n/2-\rho+j)}{\Gamma(n/2-\mu-\rho+1+j)} \right\} \end{aligned}$$

$$\times {}_2F_1 \left( \begin{matrix} -j, \mu + \rho - n/2 - j \\ n/2 - \nu - \rho + 1 \end{matrix} \middle| \frac{k^2}{q^2} \right) \}. \quad (5)$$

Let us demonstrate that the series in Eq. (5) can be expressed in terms of hypergeometric functions of two variables. To do this, let us use the following equation:

$$\sum_{j=0}^{\infty} \frac{x^j}{j!} \frac{(\alpha)_j (\beta)_j}{(\gamma)_j} {}_2F_1 \left( \begin{matrix} -j, 1 - \gamma - j \\ \delta \end{matrix} \middle| y \right) = F_4(\alpha, \beta, \gamma, \delta; x, xy), \quad (6)$$

where  $(\alpha)_j = \Gamma(\alpha + j)/\Gamma(\alpha)$ , whereas  $F_4$  is a hypergeometric function of two variables [6],

$$F_4(\alpha, \beta, \gamma, \delta; x, y) = \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \frac{(\alpha)_{m+j} (\beta)_{m+j}}{(\gamma)_m (\delta)_j} \frac{x^m}{m!} \frac{y^j}{j!}.$$

The formula (6) can be proved, for example, by expanding the hypergeometric functions in series and comparing the coefficients of different powers of  $x$  and  $y$ . Using Eq. (6), we can represent Eq. (5) in a more compact way,

$$\begin{aligned} J(\mu, \nu, \rho) &= \pi^{n/2} i^{1-n} \frac{1}{\Gamma(\mu) \Gamma(\nu) \Gamma(\rho) \Gamma(n - \mu - \nu - \rho)} \\ &\times \left\{ (p^2)^{-\mu} (k^2)^{n/2 - \nu - \rho} \Gamma(\nu + \rho - n/2) \Gamma(n/2 - \mu - \rho) \Gamma(\mu) \Gamma(n/2 - \nu) \right. \\ &\times F_4(\mu, n/2 - \nu, \mu + \rho - n/2 + 1, n/2 - \nu - \rho + 1; q^2/p^2, k^2/p^2) \\ &+ (p^2)^{-\nu} (q^2)^{n/2 - \mu - \rho} \Gamma(\mu + \rho - n/2) \Gamma(n/2 - \nu - \rho) \Gamma(\nu) \Gamma(n/2 - \mu) \\ &\times F_4(\nu, n/2 - \mu, n/2 - \mu - \rho + 1, \nu + \rho - n/2 + 1; q^2/p^2, k^2/p^2) \\ &+ (p^2)^{n/2 - \mu - \nu - \rho} \Gamma(n/2 - \nu - \rho) \Gamma(n/2 - \mu - \rho) \Gamma(\rho) \Gamma(\mu + \nu + \rho - n/2) \\ &\times F_4(\rho, \mu + \nu + \rho - n/2, \mu + \rho - n/2 + 1, \nu + \rho - n/2 + 1; q^2/p^2, k^2/p^2) \\ &+ (p^2)^{\rho - n/2} (q^2)^{n/2 - \mu - \rho} (k^2)^{n/2 - \nu - \rho} \\ &\times \Gamma(\nu + \rho - n/2) \Gamma(\mu + \rho - n/2) \Gamma(n - \mu - \nu - \rho) \Gamma(n/2 - \rho) \\ &\left. \times F_4(n - \mu - \nu - \rho, n/2 - \rho, n/2 - \mu - \rho + 1, n/2 - \nu - \rho + 1; q^2/p^2, k^2/p^2) \right\}, \quad (7) \end{aligned}$$

Let us note that in limiting cases, when  $\mu$ ,  $\nu$  or  $\rho$  vanish, Eq. (8) yields the known result. It should also be noted that, using Eq. (7) together with the formulae presented in Ref. [7], one can obtain results for the integrals of the type (1) with the momenta  $r_\alpha$ ,  $r_\alpha r_\beta, \dots$  in the numerator.

**3.** If we consider a symmetric deviation from the zero-mass shell with respect to two external legs of the corresponding Feynman diagram,  $p^2 = q^2$  (see in [7]), the expressions for the integrals can be simplified. In this case, we can use the following reduction formula:

$$\begin{aligned} F_4(\alpha, \beta, \gamma, \delta; 1, y) &= \frac{\Gamma(\gamma) \Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha) \Gamma(\gamma - \beta)} \\ &\times {}_4F_3 \left( \begin{matrix} \alpha, \beta, 1 + \alpha - \gamma, 1 + \beta - \gamma \\ \delta, (\alpha + \beta - \gamma + 2)/2, (\alpha + \beta - \gamma + 1)/2 \end{matrix} \middle| \frac{y}{4} \right). \quad (8) \end{aligned}$$

Its validity can also be demonstrated by expanding in series and comparing the coefficients. Applying Eq. (8) to Eq. (7) in the symmetric case  $p^2 = q^2$ , after some transformations we obtain

$$\begin{aligned}
J(\mu, \nu, \rho) = & \pi^{n/2} i^{1-n} \left\{ (p^2)^{n/2-\mu-\nu-\rho} \frac{\Gamma(n/2-\mu) \Gamma(n/2-\nu-\rho) \Gamma(\mu+\nu+\rho-n/2)}{\Gamma(\mu) \Gamma(\nu+\rho) \Gamma(n-\mu-\nu-\rho)} \right. \\
& \times {}_4F_3 \left( \begin{matrix} \nu, \rho, n/2-\mu, \mu+\nu+\rho-n/2 \\ (\nu+\rho)/2, (\nu+\rho+1)/2, \nu+\rho-n/2+1 \end{matrix} \middle| \frac{k^2}{4p^2} \right) \\
& + (p^2)^{-\mu} (k^2)^{n/2-\nu-\rho} \frac{\Gamma(n/2-\nu) \Gamma(n/2-\rho) \Gamma(\nu+\rho-n/2)}{\Gamma(\nu) \Gamma(\rho) \Gamma(n-\nu-\rho)} \\
& \left. \times {}_4F_3 \left( \begin{matrix} \mu, n/2-\nu, n/2-\rho, n-\mu-\nu-\rho \\ (n-\nu-\rho)/2, (n-\nu-\rho+1)/2, n/2-\nu-\rho+1 \end{matrix} \middle| \frac{k^2}{4p^2} \right) \right\}. \quad (9)
\end{aligned}$$

As expected, this expression is symmetric with respect to  $\nu$  and  $\rho$ .

It should be noted that for specific (e.g., integer) values of the parameters  $\mu$ ,  $\nu$  and  $\rho$  the result (9) can be essentially simplified. To do this, it is convenient to use the following formula:

$${}_p F_{q+1} \left( \begin{matrix} a_1, \dots, a_p, c+m \\ b_1, \dots, b_q, c \end{matrix} \middle| z \right) = \sum_{k=0}^m C_m^k \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{z^k}{(c)_k} {}_p F_q \left( \begin{matrix} a_1+k, \dots, a_p+k \\ b_1+k, \dots, b_q+k \end{matrix} \middle| z \right), \quad (10)$$

where  $C_m^k$  are the binomial coefficients. Eq. (10) shows that for integer values of  $\mu$ ,  $\nu$  and  $\rho$  the hypergeometric  ${}_4F_3$  functions in Eq. (9) can be reduced to a finite sum of the  ${}_2F_1$  functions. Expanding obtained results as  $\varepsilon = (n-4)/2 \rightarrow 0$  (which is needed for various physical applications) leads to further simplification of the results for the specific integrals.

**4.** Using the method developed in the paper, we have obtained exact results (7) and (9) for a class of the massless vertex-type integrals (1) in the framework of dimensional regularization. Since the dependence on the dimensionless combinations of the momentum variables is described in terms of the hypergeometric functions, one can use the analytic continuation formulae [6] to study various regions of the momenta.

The results obtained are of interest for the calculation of radiative corrections in gauge theories, for investigating the triangle anomalies, as well as for studying the behaviour of the vertex Green functions in the infrared region of quantum chromodynamics.

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