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Two-loop self-energy diagrams with different masses and the momentum expansion

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Abstract

An algorithm for obtaining the power series in the external momentum of two-loop self-energy diagrams with arbitrary masses of the internal particles is examined. The coefficients of the expansion are represented in terms of vacuum two-loop integrals which are calculated for general values of the masses. By comparison with a numerical calculation of some two-loop diagrams occurring in the Standard Model, it is shown that for values of the external momentum below the threshold the first few terms of the expansion provide good approximations to the complete diagrams.

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1 Introduction

When studying physically interesting problems of the Standard Model and its extensions we are often confronted with the necessity to calculate different types of Feynman loop diagrams containing particles with non-zero masses (W and Z bosons, heavy quarks, Higgs particles, etc.). We can mention the evaluation of cross sections and decay widths in different orders of perturbation theory, the examination of operator expansions and β-functions, etc. (see also the reviews [1]).

Some attention has been paid recently to the problem of evaluating massive two-loop self-energy diagrams. For some special cases (like QED corrections to the photon self-energy) this problem was considered in several publications (see, e.g., [2, 3]). The result for the corresponding diagram contains trilogarithms. On the other hand, the problem of evaluating such diagrams when all the internal particles are massive is more complicated, and exact expressions are not known. For example, in ref. [4] the result was presented in terms of a two-fold integral representation.

In the present paper we consider the expansion of two-loop massive self-energy diagrams in the external momentum. The coefficients of the expansion are calculated analytically. The paper is organized as follows. In Section 2 we construct the momentum expansion for the general case of two-loop self-energy diagrams (with five different masses) and reduce the coefficients to vacuum two-loop integrals, each of them depending on no more than three different masses only. In Section 3 we examine these vacuum integrals for the case of two different masses, while in Section 4 the general case of three masses is considered. In Section 5 we construct recurrence relations for evaluating integrals with higher powers of denominators (they are required to calculate the coefficients of the momentum expansion). In Section 6 we examine some diagrams contributing to the Z boson and photon self-energy in the Standard Model and compare the results obtained by using the momentum expansion with those obtained by a numerical program based on the algorithm in ref. [4].

2 Two-loop self-energy diagrams

All possible two-loop self-energy contributions can be reduced to scalar integrals corresponding to the two types of Feynman diagrams presented in Fig. 1 and Fig. 2. For example, the integral corresponding to Fig. 1 can be written as

$$J(\nu_1, ..., \nu_5; m_1, ..., m_5; k)$$

$$\equiv \int\int \frac{d^n p \;\; d^n q}{((k-p)^2-m_1^2)^{\nu_1}\,((k-q)^2-m_2^2)^{\nu_2}\,((p-q)^2-m_3^2)^{\nu_3}\,(p^2-m_4^2)^{\nu_4}\,(q^2-m_5^2)^{\nu_5}}, \tag{2.1}$$

where k is the external momentum (note that J really depends on k^2) and n is the space-time dimension (in the framework of dimensional regularization [5]). Here and henceforth the usual causal way of dealing with denominators in pseudo-Euclidean momentum space ($k^2 \leftrightarrow k^2 + i0$) is understood.

More simple two-loop diagrams which contain four or three internal lines, as well as products of one-loop diagrams, can be obtained from (2.1) by putting some of the powers of denominators ν_i equal to zero. Moreover, the integral corresponding to Fig. 2 can also be expressed in terms of (2.1) (for the case of integer values of ν_1 and ν_4) by use of the obvious decomposition formula

$$\frac{1}{(p^2-m_1^2)(p^2-m_4^2)} = \frac{1}{m_1^2-m_4^2}\left(\frac{1}{p^2-m_1^2} - \frac{1}{p^2-m_4^2}\right). \tag{2.2}$$

So, in general it is sufficient to consider the scalar integrals (2.1) with different powers of denominators ν_i.

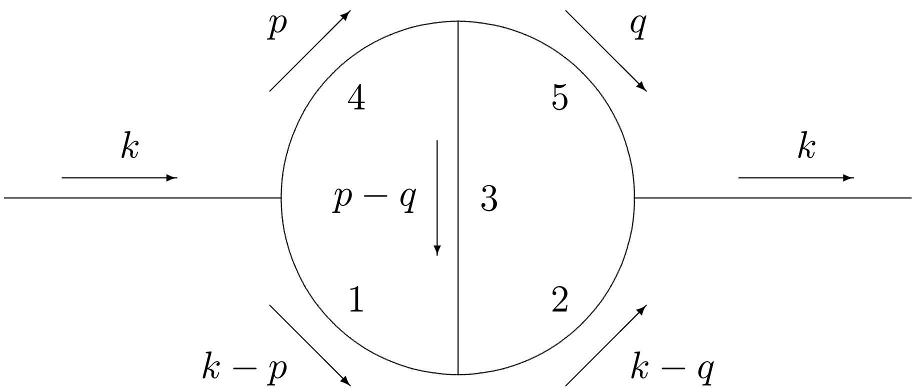

Fig. 1.

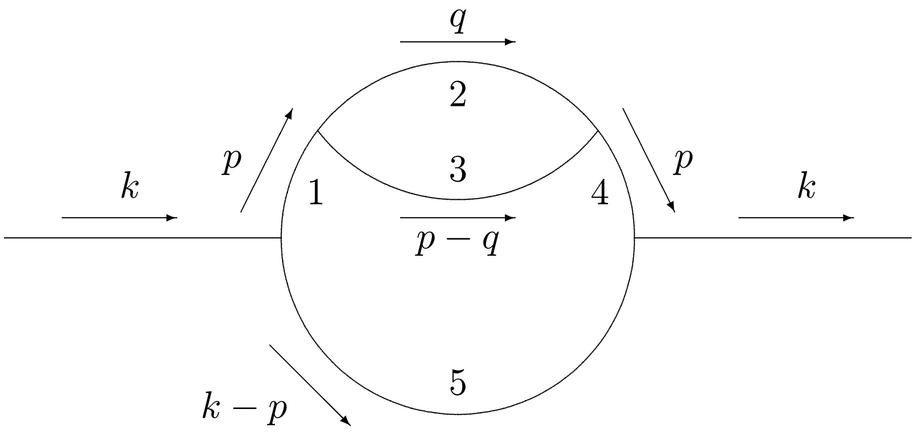

Fig. 2.

Obtaining exact expressions for the integrals (2.1) for arbitrary values of k^2 and the masses is known to be a very complicated problem. For example, in ref. [4] a double integral representation was obtained for the case $\nu_1 = ... = \nu_5 = 1$. On the other hand, in the massive case the integrals (2.1) are known to be regular functions of k^2 as $k^2 \to 0$. In the present paper we shall examine the momentum power series expansion of these integrals. To expand (2.1) with respect to k^2, it is convenient to use the momentum space d'Alembertian,

$$\Box_k \equiv \frac{\partial^2}{\partial k_\mu \partial k^\mu}. \tag{2.3}$$

For a scalar function $J(k^2)$ which is regular at $k^2 = 0$ we can derive the following expansion :

$$\begin{aligned} J(k^2) &= J(0) + \frac{k^2}{2n}\left(\Box_k J(k^2)\right)|_{k=0} + \frac{(k^2)^2}{8n(n+2)}\left(\Box_k^2 J(k^2)\right)|_{k=0} + \ldots \\ &= \sum_{j=0}^{\infty} \frac{1}{j!\,(n/2)_j}\left(\frac{k^2}{4}\right)^j \left(\Box_k^j J(k^2)\right)|_{k=0}\,, \end{aligned} \tag{2.4}$$

where

$$(a)_j \equiv \frac{\Gamma(a+j)}{\Gamma(a)} \tag{2.5}$$

is the Pochhammer symbol. Note that for the special case $m_1 = m_2 = m_4 = m_5$, $m_3 = 0$ an analogous expansion has been considered, e.g., in ref. [6] (see also [7] and references therein).

The result of applying the operator (2.3) to the integral (2.1) is (we only write shifted arguments of the integrals on the r.h.s.):

$$\begin{aligned} &\Box_k J(\nu_1,...,\nu_5; m_1,...,m_5; k) \\ &\quad = 4\left\{(\nu_1+\nu_2+1-n/2)\left(\nu_1\, J(\nu_1+1) + \nu_2\, J(\nu_2+1)\right)\right. \\ &\quad +\nu_1(\nu_1+1)m_1^2\, J(\nu_1+2) + \nu_2(\nu_2+1)m_2^2\, J(\nu_2+2) \\ &\quad \left. +\nu_1\nu_2\left((m_1^2+m_2^2-m_3^2)\, J(\nu_1+1,\nu_2+1) - J(\nu_1+1,\nu_2+1,\nu_3-1)\right)\right\}. \end{aligned} \tag{2.6}$$

For example, when we need to evaluate $\Box_k^2 J$ we apply the second $\Box_k$ to the r.h.s. of (2.6), and so on.

After applying $\Box_k^j$ we put $k = 0$. As a result, we obtain vacuum integrals (without external momentum). By using, for the cases when $m_1 \neq m_4$ or $m_2 \neq m_5$, formula (2.2) and the analogous identity for the 2nd and 5th denominators, these vacuum integrals easily can be reduced to integrals with three denominators,

$$I(\nu_1,\nu_2,\nu_3; m_1,m_2,m_3) \equiv \int\int \frac{d^n p \;\, d^n q}{(p^2-m_1^2)^{\nu_1}\,(q^2-m_2^2)^{\nu_2}\,((p-q)^2-m_3^2)^{\nu_3}}, \tag{2.7}$$

(see Fig. 3). Of course, in these integrals we may have m_4 instead of m_1 or m_5 instead of m_2. So, the coefficients of the momentum expansion (2.4) of the integrals (2.1) can be expressed in terms of vacuum integrals (2.7) with various values of ν_1, ν_2, ν_3.

When one of the indices ν_i is zero the result corresponds to the product of two massive tadpoles and can be represented in terms of gamma functions. For example:

$$I(\nu_1,\nu_2,0) = \pi^n i^{2-2n}(-m_1^2)^{n/2-\nu_1}(-m_2^2)^{n/2-\nu_2}\frac{\Gamma(\nu_1-n/2)\Gamma(\nu_2-n/2)}{\Gamma(\nu_1)\Gamma(\nu_2)}\,. \tag{2.8}$$

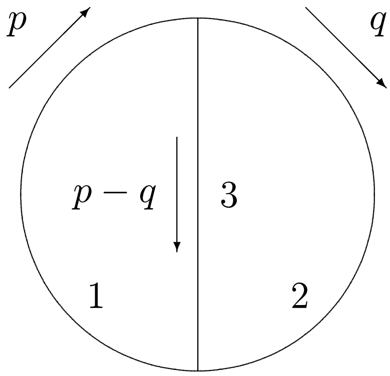

Fig. 3.

Here and below it is understood that $i^{-n}(-m^2)^{n/2} = (m^2)^{n/2}$.

Note that when we apply operator (2.3) to (2.1) we may also obtain negative values of ν_3. This means that in this case we have

$$\Big((p-q)^2 - m_3^2\Big)^{|\nu_3|} = \Big(p^2 + q^2 - m_3^2 - 2(pq)\Big)^{|\nu_3|} \tag{2.9}$$

in the numerator. The squared momenta p^2 and q^2 can be represented in terms of denominators. The integrals containing $(pq)^\alpha$ vanish for odd α's, and for even α's we use

$$\int\!\!\int \frac{d^np\ d^nq\ (pq)^{2j}}{(p^2-m_1^2)^{\nu_1}\,(q^2-m_2^2)^{\nu_2}} = \frac{(2j)!}{4^j\, j!\,(n/2)_j} \int \frac{d^np\ (p^2)^j}{(p^2-m_1^2)^{\nu_1}} \int \frac{d^nq\ (q^2)^j}{(q^2-m_2^2)^{\nu_2}}. \tag{2.10}$$

Finally, we arrive at a representation of integrals with negative ν_3 in terms of tadpole integrals (2.8). This property is well known, and we presented the formula (2.10) only for completeness.

So, the main problem in obtaining the coefficients of the expansion of (2.1) in k^2 is to evaluate the integrals (2.7) for positive integer values of ν_1, ν_2 and ν_3.

3 Vacuum integrals with two different masses

We start the examination of the vacuum integrals (2.7) with the case when two of the masses are equal: $m_1 = m_2 \equiv m,\ \ m_3 \equiv M$. This case may serve as an introduction to the methods used in this paper.

To obtain the result for the case of arbitrary values of ν_1, ν_2, ν_3 and n, it is convenient to use the method of evaluating massive Feynman integrals [8, 9] based on the Mellin-Barnes representation of massive denominators,

$$\frac{1}{(k^2-M^2)^\nu} = \frac{1}{\Gamma(\nu)}\frac{1}{2\pi i}\int\limits_{-i\infty}^{i\infty} ds\ \frac{(-M^2)^s}{(k^2)^{\nu+s}}\ \Gamma(-s)\ \Gamma(\nu+s). \tag{3.1}$$

In this formula we remember that $(k^2 \leftrightarrow k^2 + i0)$. The contours in Mellin-Barnes representations are chosen so as to separate the left and right series of poles of gamma functions occurring in the integrand.

In our case, it is sufficient to apply formula (3.1) to the third denominator only. Then the obtained vacuum integral with one massless line (with shifted power of denominator) can be evaluated in terms of gamma functions, and we arrive at the following Mellin-Barnes representation :

$$I(\nu_1,\nu_2,\nu_3;m,m,M) = \pi^n i^{2-2n} (-m^2)^{n-\nu_1-\nu_2-\nu_3} \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(n/2)}$$

$$\times \frac{1}{2\pi i} \int\limits_{-i\infty}^{i\infty} du \left(\frac{M^2}{m^2}\right)^u \Gamma(-u)\,\Gamma(n/2-\nu_3-u)$$

$$\times \frac{\Gamma(\nu_3+u)\Gamma(\nu_1+\nu_3-n/2+u)\Gamma(\nu_2+\nu_3-n/2+u)\Gamma(\nu_1+\nu_2+\nu_3-n+u)}{\Gamma(\nu_1+\nu_2+2\nu_3-n+2u)}. \tag{3.2}$$

In the following, it will be convenient to use the dimensionless variable

$$z \equiv \frac{M^2}{4m^2}\ . \tag{3.3}$$

For $|z|<1$ we close the integration contour in (3.2) to the right and obtain

$$I(\nu_1,\nu_2,\nu_3;m,m,M) = \pi^n i^{2-2n} (-m^2)^{n-\nu_1-\nu_2-\nu_3}$$

$$\times \left\{ \frac{\Gamma(n/2-\nu_3)\Gamma(\nu_1+\nu_3-n/2)\Gamma(\nu_2+\nu_3-n/2)\Gamma(\nu_1+\nu_2+\nu_3-n)}{\Gamma(n/2)\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_1+\nu_2+2\nu_3-n)} \right.$$

$$\times {}_4F_3\left(\left. \begin{array}{c} \nu_3,\ \nu_1+\nu_3-n/2,\ \nu_2+\nu_3-n/2,\ \nu_1+\nu_2+\nu_3-n \\ (\nu_1+\nu_2+2\nu_3-n)/2,\ (\nu_1+\nu_2+2\nu_3-n+1)/2,\ \nu_3-n/2+1 \end{array} \right| z \right)$$

$$+ (4z)^{n/2-\nu_3} \frac{\Gamma(\nu_3-n/2)\Gamma(\nu_1+\nu_2-n/2)}{\Gamma(\nu_3)\Gamma(\nu_1+\nu_2)}$$

$$\left. \times {}_4F_3\left(\left. \begin{array}{c} \nu_1,\ \nu_2,\ \nu_1+\nu_2-n/2,\ n/2 \\ (\nu_1+\nu_2)/2,\ (\nu_1+\nu_2+1)/2,\ n/2-\nu_3+1 \end{array} \right| z \right) \right\}, \tag{3.4}$$

where we have used the standard notation for the generalized hypergeometric function of one variable (see, e.g., [10]),

$${}_PF_Q\left(\left. \begin{array}{c} a_1,\ \dots\ a_P \\ c_1,\ \dots\ c_Q \end{array} \right| z \right) \equiv \sum_{j=0}^{\infty} \frac{z^j}{j!} \frac{(a_1)_j \dots (a_P)_j}{(c_1)_j \dots (c_Q)_j}, \tag{3.5}$$

where $(a)_j$ is the Pochhammer symbol (2.5).

In the case $\nu_1=\nu_2=\nu_3=1$ the ${}_4F_3$ functions reduce to ${}_2F_1$ functions. Expanding them in $\varepsilon=(4-n)/2$ and keeping the singular and $O(1)$ terms only, we get

$$
\begin{aligned}
I(1,1,1;m,m,M) &= \pi^{4-2\varepsilon}(m^2)^{1-2\varepsilon}\, A(\varepsilon) \\
&\times \left\{-\frac{1}{\varepsilon^2}(1+2z)+\frac{1}{\varepsilon}(4z\ln(4z)) - 2z\ln^2(4z) + 2(1-z)\Phi(z)\right\}
\end{aligned}
\tag{3.6}
$$

where

$$
\begin{aligned}
\Phi(z) &= 4z\left[(2-\ln(4z))\ {}_2F_1\left(\left.\begin{array}{c}1,1\\3/2\end{array}\right| z\right)\right. \\
&\left. -\partial_a\ {}_2F_1\left(\left.\begin{array}{c}1,1\\3/2\end{array}\right| z\right) - \partial_c\ {}_2F_1\left(\left.\begin{array}{c}1,1\\3/2\end{array}\right| z\right)\right],
\end{aligned}
\tag{3.7}
$$

$$
\begin{aligned}
A(\varepsilon) &\equiv \frac{\Gamma^2(1+\varepsilon)}{(1-\varepsilon)(1-2\varepsilon)} \\
&= 1+\varepsilon(3-2\gamma)+\varepsilon^2\left(7-6\gamma+2\gamma^2+\frac{\pi^2}{6}\right)+O(\varepsilon^3),
\end{aligned}
\tag{3.8}
$$

and $\gamma = 0.57721566...$ is Euler's constant. In formula (3.7) we used the following notation for the derivatives of the hypergeometric function with respect to the parameters a and c:

$$
\begin{aligned}
\partial_a\ {}_2F_1\left(\left.\begin{array}{c}a,b\\c\end{array}\right| z\right) &\equiv \frac{\partial}{\partial a}\ {}_2F_1\left(\left.\begin{array}{c}a,b\\c\end{array}\right| z\right) \\
&= \sum_{j=0}^{\infty}\frac{z^j}{j!}\frac{(a)_j\ (b)_j}{(c)_j}(\psi(a+j)-\psi(a)),
\end{aligned}
\tag{3.9}
$$

$$
\begin{aligned}
\partial_c\ {}_2F_1\left(\left.\begin{array}{c}a,b\\c\end{array}\right| z\right) &\equiv \frac{\partial}{\partial c}\ {}_2F_1\left(\left.\begin{array}{c}a,b\\c\end{array}\right| z\right) \\
&= -\sum_{j=0}^{\infty}\frac{z^j}{j!}\frac{(a)_j\ (b)_j}{(c)_j}(\psi(c+j)-\psi(c)),
\end{aligned}
\tag{3.10}
$$

where $\psi(a) \equiv (d/da)\ln(\Gamma(a))$. The combinations of ψ functions of integer and half-integer arguments occurring on the r.h.s.'s of (3.9) and (3.10) (for $a=b=1$, $c=3/2$) are known rational numbers. Many useful properties of the functions (3.9) and (3.10) (integral representations, analytic continuation formulae, etc.) can be obtained by differentiating appropriate formulae for ${}_2F_1$ with respect to parameters. Note that the same functions as in (3.7) have occurred in ref. [11] where the QCD radiative correction to Higgs decay into two quarks was considered.

By using a parametric integral representation for the ${}_2F_1$ function and its parametric derivatives it is easy to obtain the following results (for $0 \le z < 1$):

$$
{}_2F_1\left(\begin{array}{c} 1,1 \\ 3/2 \end{array} \middle| z \right) = \frac{\arcsin\sqrt{z}}{\sqrt{z(1-z)}} , \tag{3.11}
$$

$$
\begin{aligned}
&\partial_a \, {}_2F_1\left(\begin{array}{c} 1,1 \\ 3/2 \end{array} \middle| z \right) + \partial_c \, {}_2F_1\left(\begin{array}{c} 1,1 \\ 3/2 \end{array} \middle| z \right) - 2 \, {}_2F_1\left(\begin{array}{c} 1,1 \\ 3/2 \end{array} \middle| z \right) \\
&= -\frac{1}{\sqrt{z(1-z)}} \left\{ \ln(4z) \arcsin\sqrt{z} + \mathrm{Cl}_2\left(2\arcsin\sqrt{z} \right) \right\} ,
\end{aligned} \tag{3.12}
$$

where Cl_2 is Clausen's integral function (see, e.g., [12]),

$$
\mathrm{Cl}_2\,(\theta) = -\int\limits_0^\theta d\theta \;\ln\left| 2\sin\frac{\theta}{2} \right| . \tag{3.13}
$$

As a result, we get

$$
\Phi(z) = 4\sqrt{\frac{z}{1-z}}\; \mathrm{Cl}_2\left(2\arcsin\sqrt{z} \right) . \tag{3.14}
$$

It should be noted that an analogous representation (in terms of Lobachevskiy's function) has been obtained in ref. [13].

In particular, if $M = m$ $(z = 1/4)$ we have

$$
I(1,1,1;m,m,m) = \pi^{4-2\varepsilon} (m^2)^{1-2\varepsilon} A(\varepsilon) \left\{ -\frac{3}{2\varepsilon^2} + \frac{27}{2} S_2 \right\} \tag{3.15}
$$

with (we follow the notation of ref. [3])

$$
S_2 = \frac{4}{9\sqrt{3}} \mathrm{Cl}_2\left(\frac{\pi}{3} \right) = 0.2604341... \tag{3.16}
$$

where $\mathrm{Cl}_2\,(\pi/3) = 1.0149417...$ corresponds to the maximum of Clausen's integral [12] and cannot be represented in terms of other known transcendental constants. This constant has appeared before in two-loop massive calculations (see, e.g., [14, 3, 13]). Note that in ref. [15] results for the integrals (2.7) have been obtained in the form of hypergeometric series which follow from (3.4) if we take $M = m$ $(z = 1/4)$. After extraction of singularities in ε, these expressions are numerical series. A numerical comparison shows that they coincide with the results expressed in terms of (3.16).

To obtain, from (3.7), the result in the region $z > 1$ we may use the analytic continuation formulae from z to $(1-z)$ (see, e.g., [11]):

$$
{}_2F_1\left(\begin{array}{c} 1,1 \\ 3/2 \end{array} \middle| 1-z \right) = \frac{\pi}{2} \frac{1}{\sqrt{z(1-z)}} - {}_2F_1\left(\begin{array}{c} 1,1 \\ 3/2 \end{array} \middle| z \right) , \tag{3.17}
$$

$$\partial_a\, {}_2F_1\left(\begin{array}{c} 1,1 \\ 3/2 \end{array} \middle| 1-z \right) = -\frac{\pi}{2}\frac{\ln(4z)}{\sqrt{z(1-z)}}$$

$$+2\, {}_2F_1\left(\begin{array}{c} 1,1 \\ 3/2 \end{array} \middle| z \right) - \partial_a\, {}_2F_1\left(\begin{array}{c} 1,1 \\ 3/2 \end{array} \middle| z \right) - \partial_c\, {}_2F_1\left(\begin{array}{c} 1,1 \\ 3/2 \end{array} \middle| z \right). \qquad (3.18)$$

Formula (3.18) can be obtained by differentiation of the usual analytic continuation formula for ${}_2F_1$ with respect to the parameter a. Using these formulae we get

$$\Phi(z) = 4z\left[\ln(4z)\, {}_2F_1\left(\begin{array}{c} 1,1 \\ 3/2 \end{array} \middle| 1-z \right) + \partial_a\, {}_2F_1\left(\begin{array}{c} 1,1 \\ 3/2 \end{array} \middle| 1-z \right)\right]. \qquad (3.19)$$

This formula gives us the expansion near $z=1$.

The region $z>1$ corresponds to negative values of the arguments of the hypergeometric functions in (3.19). In this case the function (3.17) can be represented in terms of logarithms while the function (3.18) contains a dilogarithm (Spence function) [12], defined by

$$\mathrm{Li}_2\,(\xi) = -\int\limits_0^1 dt \frac{\ln(1-\xi t)}{t}. \qquad (3.20)$$

Finally, we obtain from (3.19)

$$\Phi(z) = \frac{1}{\lambda}\left[-4\,\mathrm{Li}_2\left(\frac{1-\lambda}{2}\right) + 2\ln^2\left(\frac{1-\lambda}{2}\right) - \ln^2(4z) + \frac{\pi^2}{3}\right], \qquad (3.21)$$

where

$$\lambda(z) = \sqrt{1-\frac{1}{z}}. \qquad (3.22)$$

So, we have representations of $I(1,1,1;m,m,M)$ for all values of the masses.

4 Vacuum integrals with three different masses

Let us consider now the general case of (2.7) when all three masses m_1, m_2 and m_3 are different. Using the Mellin-Barnes representation for massive denominators yields

$$I(\nu_1,\nu_2,\nu_3;m_1,m_2,m_3) = \pi^n i^{2-2n}(-m_3^2)^{n-\nu_1-\nu_2-\nu_3}\frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(n/2)}$$

$$\times\frac{1}{(2\pi i)^2}\int\limits_{-i\infty}^{i\infty}\int\limits_{-i\infty}^{i\infty} ds\, dt\, x^s\, y^t \Gamma(-s)\Gamma(-t)\Gamma(n/2-\nu_1-s)\Gamma(n/2-\nu_2-t)$$

$$\times\Gamma(\nu_1+\nu_2-n/2+s+t)\Gamma(\nu_1+\nu_2+\nu_3-n+s+t)\,. \qquad (4.1)$$

where the two dimensionless variables

$$x \equiv \frac{m_1^2}{m_3^2} \quad \text{and} \quad y \equiv \frac{m_2^2}{m_3^2} \qquad (4.2)$$

are used. Closing the contours to the right we get (for arbitrary ν_1, ν_2, ν_3 and n)

$$I(\nu_1,\nu_2,\nu_3;m_1,m_2,m_3) = \pi^n i^{2-2n} (-m_3^2)^{n-\nu_1-\nu_2-\nu_3} \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(n/2)}$$

$$\times \Big\{ \Gamma(n/2-\nu_1)\Gamma(n/2-\nu_2)\Gamma(\nu_1+\nu_2-n/2)\Gamma(\nu_1+\nu_2+\nu_3-n)$$

$$\times F_4\left(\nu_1+\nu_2+\nu_3-n,\ \nu_1+\nu_2-n/2;\ \nu_1-n/2+1,\ \nu_2-n/2+1 \,|\, x,\ y\right)$$

$$+y^{n/2-\nu_2}\Gamma(n/2-\nu_1)\Gamma(\nu_2-n/2)\Gamma(\nu_1)\Gamma(\nu_1+\nu_3-n/2)$$

$$\times F_4\left(\nu_1,\ \nu_1+\nu_3-n/2;\ \nu_1-n/2+1,\ n/2-\nu_2+1 \,|\, x,\ y\right)$$

$$+x^{n/2-\nu_1}\Gamma(\nu_1-n/2)\Gamma(n/2-\nu_2)\Gamma(\nu_2)\Gamma(\nu_2+\nu_3-n/2)$$

$$\times F_4\left(\nu_2,\ \nu_2+\nu_3-n/2;\ n/2-\nu_1+1,\ \nu_2-n/2+1 \,|\, x,\ y\right)$$

$$+x^{n/2-\nu_1}\ y^{n/2-\nu_2}\Gamma(\nu_1-n/2)\Gamma(\nu_2-n/2)\Gamma(\nu_3)\Gamma(n/2)$$

$$\times F_4\left(\nu_3,\ n/2;\ n/2-\nu_1+1,\ n/2-\nu_2+1 \,|\, x,\ y\right)\Big\}\ , \tag{4.3}$$

where

$$F_4\left(a,b;c,d|x,y\right) = \sum_{j=0}^{\infty}\sum_{l=0}^{\infty} \frac{x^j\ y^l}{j!\ l!}\ \frac{(a)_{j+l}\ (b)_{j+l}}{(c)_j\ (d)_l} \tag{4.4}$$

is Appell's hypergeometric function of two variables (see, e.g., [10]) and $(a)_j$ is defined by (2.5). Formula (4.3) gives us the expansion in x and y for the case $\sqrt{x}+\sqrt{y} \le 1$ (this corresponds to the region of convergence of the F_4 functions).

For the important special case, $\nu_1=\nu_2=\nu_3=1$ ($n \equiv 4-2\varepsilon$), we can use known reduction formulae for F_4-functions (see, e.g., [16, 17]). As a result, we find

$$I(1,1,1;m_1,m_2,m_3) = \pi^{4-2\varepsilon}(m_3^2)^{1-2\varepsilon}\frac{1}{\Gamma(2-\varepsilon)}$$

$$\times \Bigg\{ \Gamma^2(1-\varepsilon)\Gamma(\varepsilon)\Gamma(-1+2\varepsilon)\left(\frac{1-wz}{(1-w)(1-z)}\right)^{1-2\varepsilon}$$

$$+\Gamma(1-\varepsilon)\Gamma(-1+\varepsilon)\Gamma(\varepsilon)\left(\frac{-z}{(1-w)(1-z)}\right)^{1-\varepsilon}$$

$$\times(1-w)\ {}_2F_1\left(\begin{array}{c} \varepsilon,\ 1 \\ 2-\varepsilon \end{array}\middle|\ -\frac{z(1-w)}{1-z}\right)$$

$$+\Gamma(1-\varepsilon)\Gamma(-1+\varepsilon)\Gamma(\varepsilon)\left(\frac{-w}{(1-w)(1-z)}\right)^{1-\varepsilon}$$

$$\times(1-z)\ {}_2F_1\left(\begin{array}{c} \varepsilon,\ 1 \\ 2-\varepsilon \end{array}\middle|\ -\frac{w(1-z)}{1-w}\right)$$

$$+\Gamma(2-\varepsilon)\Gamma^2(-1+\varepsilon)\left(\frac{-z}{(1-w)(1-z)}\right)^{1-\varepsilon}\left(\frac{-w}{(1-w)(1-z)}\right)^{1-\varepsilon}$$

$$\times(1-w)\ (1-z)\ {}_2F_1\left(\left.\begin{array}{c}\varepsilon,\ 1\\ 2-\varepsilon\end{array}\right| wz\right)\Bigg\}, \tag{4.5}$$

where

$$w=\frac{1}{2y}\left(-1+x+y+\lambda\right)\ ,\ z=\frac{1}{2x}\left(-1+x+y+\lambda\right)\ , \tag{4.6}$$

with

$$\lambda(x,y)\equiv\sqrt{(1-x-y)^2-4xy}\ . \tag{4.7}$$

This formula can be connected with the result obtained in [18] by using analytic continuation formulae that express ${}_2F_1$ functions of ξ in terms of ${}_2F_1$ functions of $1-\xi$. By use of the parametric integral representation for the ${}_2F_1$ function occurring in (4.5) it is easy to obtain the following expansion as $\varepsilon\to 0$ (keeping terms up to $O(\varepsilon^2)$ only):

$${}_2F_1\left(\left.\begin{array}{c}\varepsilon,\ 1\\ 2-\varepsilon\end{array}\right|\xi\right)\simeq\frac{1-\varepsilon}{1-2\varepsilon}\left\{1+\frac{1-\xi}{\xi}\left(\varepsilon\ln\left(1-\xi\right)-\varepsilon^2\left(\ln^2\left(1-\xi\right)+\mathrm{Li}_2\left(\xi\right)\right)\right)\right\}. \tag{4.8}$$

Using this expansion and some well-known relations for dilogarithms [12] we get the following result for the integral with $\nu_1=\nu_2=\nu_3=1$ (as $n\to 4$):

$$I(1,1,1;m_1,m_2,m_3)=\pi^{4-2\varepsilon}(m_3^2)^{1-2\varepsilon}\frac{A(\varepsilon)}{2}$$

$$\times\left\{-\frac{1}{\varepsilon^2}(1+x+y)+\frac{2}{\varepsilon}(x\ln x+y\ln y)\right.$$

$$\left.-x\ln^2 x-y\ln^2 y+(1-x-y)\ln x\ln y-\lambda^2\Phi(x,y)\right\}\ , \tag{4.9}$$

where

$$\Phi(x,y)\ =\ \frac{1}{\lambda}\left\{2\ln\left(\frac{1+x-y-\lambda}{2}\right)\ln\left(\frac{1-x+y-\lambda}{2}\right)-\ln x\ln y\right.$$

$$\left.-2\,\mathrm{Li}_2\left(\frac{1+x-y-\lambda}{2}\right)-2\,\mathrm{Li}_2\left(\frac{1-x+y-\lambda}{2}\right)+\frac{\pi^2}{3}\right\}\ . \tag{4.10}$$

The functions $\lambda(x,y)$ and $\Phi(x,y)$ coincide with $\lambda(z)$ (3.22) and $\Phi(z)$ (3.21) when $x=y=1/(4z)$.

Note that these results were obtained in the region $\lambda^2\ge 0$ and $\sqrt{x}+\sqrt{y}\le 1$. By permutation of m_1,m_2,m_3 we can also obtain results for the region $\sqrt{x}-\sqrt{y}\ge 1$ and the region $\sqrt{y}-\sqrt{x}\ge 1$.

It is interesting to note that the function $\Phi(x,y)$ (4.10) is the same as in the case of the massless triangle diagram (see, e.g., [19]) where x and y are constructed from external momenta squared instead of masses. So, all results for $\Phi(x,y)$ obtained in the present paper are also applicable to that case.

The analytic continuation to the region $\lambda^2 < 0$ can be done by use of the quadratic transformation,

$$ {}_2F_1\left(\begin{array}{c} a,\ b \\ a-b+1 \end{array} \middle| \xi \right) = (1-\xi)^{-a}\ {}_2F_1\left(\begin{array}{c} a/2,\ (a+1)/2-b \\ a-b+1 \end{array} \middle| -\frac{4\xi}{(1-\xi)^2} \right), \quad (4.11) $$

which we apply to the ${}_2F_1$ functions occurring in (4.5). Then the transition to inverse arguments yields

$$ \begin{aligned} I(1,1,1;m_1,m_2,m_3) &= \pi^{4-2\varepsilon}(m_3^2)^{1-2\varepsilon}A(\varepsilon) \\ &\times\frac{1}{2\varepsilon^2}\left\{ x^{-\varepsilon}y^{-\varepsilon}(1-x-y)\ {}_2F_1\left(\begin{array}{c} \varepsilon,\ 1 \\ 1/2+\varepsilon \end{array} \middle| -\frac{\lambda^2}{4xy} \right) \right. \\ &-x^{-\varepsilon}(1+x-y)\ {}_2F_1\left(\begin{array}{c} \varepsilon,\ 1 \\ 1/2+\varepsilon \end{array} \middle| -\frac{\lambda^2}{4x} \right) \\ &\left. -y^{-\varepsilon}(1-x+y)\ {}_2F_1\left(\begin{array}{c} \varepsilon,\ 1 \\ 1/2+\varepsilon \end{array} \middle| -\frac{\lambda^2}{4y} \right) \right\}. \end{aligned} \quad (4.12) $$

This formula can be used to examine the behaviour of $I(1,1,1;m_1,m_2,m_3)$ near $\lambda^2 = 0$. It is valid for arbitrary n because we did not expand in ε. Using the definition of x and y (4.2) we see that it is completely symmetric in m_1, m_2, m_3. However, it is only valid outside the region bounded by the lines $1-x-y=0$ $(0<x<1)$, $1+x-y=0$ $(x>0)$ and $1-x+y=0$ $(x>1)$, where the arguments of the corresponding ${}_2F_1$ functions are equal to one. A careful examination of the ${}_2F_1$ functions near these lines shows that if we write an additional term in the braces in (4.12),

$$ \left\{ \dots \right\} \to \left\{ \dots + 2\pi\varepsilon\frac{\Gamma(1+2\varepsilon)}{\Gamma^2(1+\varepsilon)}(-\lambda^2)^{1/2-\varepsilon}\theta(x+y-1)\theta(1-x+y)\theta(1+x-y) \right\}, \quad (4.13) $$

we obtain an expression which is valid for all positive values of x and y. If we consider $\varepsilon \to 0$ the following expansion up to ε^2 terms can be derived:

$$ \begin{aligned} &{}_2F_1\left(\begin{array}{c} \varepsilon,\ 1 \\ 1/2+\varepsilon \end{array} \middle| \eta \right) \simeq 1 + 2\varepsilon\eta\ {}_2F_1\left(\begin{array}{c} 1,\ 1 \\ 3/2 \end{array} \middle| \eta \right) \\ &+2\varepsilon^2\eta\left\{ -2\ {}_2F_1\left(\begin{array}{c} 1,\ 1 \\ 3/2 \end{array} \middle| \eta \right) + \partial_a\ {}_2F_1\left(\begin{array}{c} 1,\ 1 \\ 3/2 \end{array} \middle| \eta \right) + \partial_c\ {}_2F_1\left(\begin{array}{c} 1,\ 1 \\ 3/2 \end{array} \middle| \eta \right) \right\}. \end{aligned} \quad (4.14) $$

Here we obtained the same functions as in equation (3.7) for the case of two different masses.

In this case, in the representations (3.11) and (3.12) it is more convenient to use arccos rather than arcsin in the arguments of Clausen's function. So, we get the following representation for the function $\Phi(x,y)$ in the region $\lambda^2 \le 0$ ($\sqrt{x}+\sqrt{y} \ge 1$).

$$ \begin{aligned} \Phi(x,y) \quad = \quad & \frac{2}{\sqrt{-\lambda^2}}\left\{ \mathrm{Cl}_2\left(2\arccos\left(\frac{-1+x+y}{2\sqrt{xy}} \right) \right) + \mathrm{Cl}_2\left(2\arccos\left(\frac{1+x-y}{2\sqrt{x}} \right) \right) \right. \\ & \left. + \mathrm{Cl}_2\left(2\arccos\left(\frac{1-x+y}{2\sqrt{y}} \right) \right) \right\}. \end{aligned} \quad (4.15) $$

Note that the result (4.9), (4.15) is also completely symmetric. This expression can be used inside the parabola which continues the curve $\sqrt{x}+\sqrt{y}=1$. The analogous representation was obtained in ref. [20] in terms of Lobachevskiy's function.

5 Recurrence relations

When we calculate the coefficients of the expansion (2.4) we also need to evaluate integrals (2.7) with higher powers of denominators. Instead of working out eqs. (3.4) and (4.3), we will use recurrence relations. To derive them, it is convenient to use the integration-by-parts technique [21]; for example:

$$\int\int d^np\, d^nq \frac{\partial}{\partial p_\mu}\left\{\frac{p_\mu}{(p^2-m_1^2)^{\nu_1}\,(q^2-m_2^2)^{\nu_2}\,((p+q)^2-m_3^2)^{\nu_3}}\right\}=0. \qquad (5.1)$$

Identities of the type of (5.1) make it possible to construct a recursive procedure for evaluating $I(\nu_1,\nu_2,\nu_3)$ (in this section we omit the arguments m_1,m_2,m_3) with integer ν's (by analogy with [19]).

For the integrals (2.7), we get three independent conditions. The determinant of the corresponding system of equations for $I(\nu_1+1,\nu_2,\nu_3)$, $I(\nu_1,\nu_2+1,\nu_3)$ and $I(\nu_1,\nu_2,\nu_3+1)$ is

$$\Delta(m_1^2,m_2^2,m_3^2)=2(m_1^2m_2^2+m_1^2m_3^2+m_2^2m_3^2)-(m_1^4+m_2^4+m_3^4)=-m_3^4\lambda^2(x,y) \qquad (5.2)$$

where $\lambda(x,y)$ is defined by (4.7). Solving these equations yields

$$\begin{aligned}
I(\nu_1+1,\nu_2,\nu_3) &= \frac{1}{\nu_1 m_1^2\Delta(m_1^2,m_2^2,m_3^2)} \\
&\times\Big\{\Big[\nu_2(m_1^2-m_3^2)(m_1^2-m_2^2+m_3^2)+\nu_3(m_1^2-m_2^2)(m_1^2+m_2^2-m_3^2) \\
&\qquad +nm_1^2(-m_1^2+m_2^2+m_3^2)-\nu_1\Delta(m_1^2,m_2^2,m_3^2)\Big]\,I(\nu_1,\nu_2,\nu_3) \\
&\quad +\nu_2m_2^2(m_1^2-m_2^2+m_3^2)\left[I(\nu_1,\nu_2+1,\nu_3-1)-I(\nu_1-1,\nu_2+1,\nu_3)\right] \\
&\quad +\nu_3m_3^2(m_1^2+m_2^2-m_3^2)\left[I(\nu_1,\nu_2-1,\nu_3+1)-I(\nu_1-1,\nu_2,\nu_3+1)\right]\Big\} \qquad (5.3)
\end{aligned}$$

and expressions for $I(\nu_1,\nu_2+1,\nu_3)$ and $I(\nu_1,\nu_2,\nu_3+1)$ (which can be obtained from (5.3) by permutation of indices). These results make it possible to evaluate integrals with $\nu_1+\nu_2+\nu_3=\sigma+1$ in terms of the integrals with $\nu_1+\nu_2+\nu_3=\sigma$. It can also be noted that there is an additional condition for integrals with the same sum $\nu_1+\nu_2+\nu_3$, which we have used to reduce the number of terms on the r.h.s. of (5.3),

$$\begin{aligned}
&\Big[\nu_1(m_2^2-m_3^2)+\nu_2(m_3^2-m_1^2)+\nu_3(m_1^2-m_2^2)\Big]\,I(\nu_1,\nu_2,\nu_3) \\
&=\nu_1m_1^2\left[I(\nu_1+1,\nu_2-1,\nu_3)-I(\nu_1+1,\nu_2,\nu_3-1)\right] \\
&\quad+\nu_2m_2^2\left[I(\nu_1,\nu_2+1,\nu_3-1)-I(\nu_1-1,\nu_2+1,\nu_3)\right] \\
&\quad+\nu_3m_3^2\left[I(\nu_1-1,\nu_2,\nu_3+1)-I(\nu_1,\nu_2-1,\nu_3+1)\right]. \qquad (5.4)
\end{aligned}$$

Formula (5.3) (and permutations) enable one to evaluate integrals with any positive integer ν's in terms of $I(1,1,1)$ and trivial boundary integrals (2.8) (when one of ν's is equal to zero). These recurrence relations can easily be algorithmized (to do this, we have used the REDUCE system [22]). As an example, it is easy to check that the result for $I(2,1,1)$ (obtained by applying relation (5.3) in the case $\nu_1 = \nu_2 = \nu_3 = 1$) after some transformations coincides with the results presented in refs. [14, 23] in terms of dilogarithms.

In the case $m_1 = m_2 \equiv m, \;\; m_3 \equiv M \;\; (z \equiv M^2/(4m^2))$ we get (at $\varepsilon = 0$)

$$I(2,2,1) = \frac{\pi^4}{2m^2}\frac{1}{(1-z)}\left\{(1-2z)\frac{\mathrm{Cl}_2\left(2\arcsin\sqrt{z}\right)}{\sqrt{z(1-z)}} + \ln(4z)\right\}, \quad z \le 1, \tag{5.5}$$

$$\begin{aligned} I(2,2,1) = \frac{\pi^4}{2m^2}\frac{1}{(1-z)}\Bigg\{ & \ln(4z) \\ & + \frac{2z-1}{\lambda z}\left[\mathrm{Li}_2\left(\frac{1-\lambda}{2}\right) - \frac{1}{2}\ln^2\left(\frac{1-\lambda}{2}\right) + \frac{1}{4}\ln^2(4z) - \frac{\pi^2}{12}\right]\Bigg\}, \quad z \ge 1. \end{aligned} \tag{5.6}$$

These results give the first term of expansion (2.4) for the integral corresponding to the diagram in Fig. 1 with $m_1 = m_2 = m_4 = m_5 = m, \; m_3 = M$. Note that expressions (5.5) and (5.6) coincide at $z = 1$ and yield

$$I(2,2,1) = \frac{\pi^4}{3m^2}(4\ln 2 - 1), \qquad z = 1 \qquad (M = 2m)\,. \tag{5.7}$$

6 Numerical results

In this section we shall demonstrate our method by considering the integral

$$J(m_1, m_2, m_3, m_4, m_5; k) \equiv J(1,1,1,1,1; m_1, m_2, m_3, m_4, m_5; k). \tag{6.1}$$

It is finite and it can be calculated for arbitrary masses by a method described in ref. [4] which involves a two dimensional numerical integration. We shall approximate the integral (6.1) by taking the first few terms of its momentum expansion, and then compare the results with the values we obtain by using the method in ref. [4].

In our first example we take $m_1 = m_2 = m_4 = m_5 = m$ and $m_3 = M$ and write the momentum expansion as:

$$J(m,m,M,m,m;k) = \frac{\pi^4}{m^2}\sum_{j=0}^{\infty} c_j(z)\left(\frac{k^2}{4m^2}\right)^j, \tag{6.2}$$

with $z = M^2/(4m^2)$. By performing the procedure described in the previous sections we obtain expressions for the coefficients $c_j(z)$ that contain z, $\ln z$ and $\Phi(z)$. In the denominator they contain powers of z and $(1-z)$, that are introduced when the recurrence relation (5.3) is used. To avoid numerical instabilities, we derive expansions of our expressions in z and in $(z-1)$ which we use when z is close to 0 or

1. These expansions can easily be obtained by using the expressions for Φ in terms of hypergeometric functions (3.7) and (3.19). Near $z = 1$ the coefficients $c_j(z)$ are analytic. The expansions near $z = 0$ contain powers of z, and in addition terms of the form $z^l \ln z$. As can be seen from Fig. 4 for the case of $c_0(z)$, the expansions can give good approximations to the exact coefficients in a wide range of z values.

The first six coefficients $c_j(z)$ are plotted in Fig. 5. Their exact values are listed in Table 1 for three special values of z. For $z = 0$ they coincide with the results given in ref. [3].

	$z = 0$	$z = \frac{1}{4}$ $(M = m)$	$z = 1$ $(M = 2m)$
$c_0(z)$	1	$3S_2$	$\frac{4\ln 2-1}{3}$
$c_1(z)$	$\frac{13}{18}$	$\frac{2(4+9S_2)}{27}$	$\frac{32\ln 2-1}{70}$
$c_2(z)$	$\frac{388}{675}$	$\frac{4(29-36S_2)}{243}$	$\frac{4(1992\ln 2+83)}{31185}$
$c_3(z)$	$\frac{5309}{11025}$	$\frac{8(367-990S_2)}{3645}$	$\frac{7680\ln 2+529}{45045}$
$c_4(z)$	$\frac{206624}{496125}$	$\frac{16(317179-1063440S_2)}{3444525}$	$\frac{32(1417840\ln 2+113193)}{363738375}$
$c_5(z)$	$\frac{13260704}{36018675}$	$\frac{32(566101-2061360S_2)}{6200145}$	$\frac{1312(73920\ln 2+6221)}{1003917915}$

Table 1. The first coefficients c_j for $z = 0$, $z = 1/4$ and $z = 1$. The constant S_2 is defined in (3.16).

Fig. 6 shows the first terms of the momentum expansion (6.2), and also the value of J obtained by numerical integration, in the region below the threshold at $k^2 = 4m^2$ (the right edge of the plot). In this example we chose $M = M_Z$ and $m = m_t$, the masses of the Z boson and the top quark. For the latter we picked a value which is in the middle of the allowed range of masses. This combination of masses occurs in two-loop corrections to the Z and photon propagators in the Standard Model. In this particular case only three terms are needed to obtain an accuracy of 1% when $k^2/m_t^2 = 1$. To obtain 1% accuracy when $k^2/m_t^2 = 2$, five terms are needed. When $k^2/m_t^2 = 3$, the sum of the first six terms differs about 5% from the numerical value. These numbers depend on z, but the picture for other values of z is very similar. Notice that if $k^2 \approx M_Z^2$, $k^2/m_t^2 \approx 0.42$, which is still in the region where the momentum expansion approximation works well.

An example involving three different masses is shown in Fig. 7, where $m_1 = m_4 = m_t$, $m_2 = m_5 = M_W$ (the W boson mass) and $m_3 = m_b$ (the b quark mass). In this case the first threshold is at $k^2 = 4M_W^2$. If we choose $k^2 \approx M_Z^2$, $k^2/M_W^2 \approx 1.3$, so we are again well below the threshold.

7 Summary and conclusions

In this paper we considered an algorithm to construct the expansion in the momentum squared for the general case of massive two-loop propagator-type diagrams. We showed that, by using (2.6), the coefficients of the expansion (2.4) can be expressed in terms of vacuum two-loop integrals (2.7), each of them involving no more than three different masses. Then we calculated these integrals and obtained representations for all regions of the values of the masses. By use of the integration-by-parts technique we constructed a recursive procedure to evaluate the integrals (2.7) with higher powers of the denominators required for the coefficients of the expansion (2.4). In this way, all the coefficients can be expressed in terms of the function $\Phi(x, y)$ involving dilogarithms (4.10) or Clausen functions (4.15), and powers and logarithms of the masses.

Note that the formulae we use ((2.4), (2.6), (5.3), etc.) are valid for arbitrary n. Therefore, the algorithm can also be applied to ultraviolet divergent integrals that occur in realistic calculations. In this case, we get poles in ε which should be cancelled by renormalization counterterms.

By comparison with the results of a numerical program for evaluating two-loop diagrams (based on a two-fold integral representation [4]) for some physically interesting cases, we checked that our algorithm provides a rapidly converging series in the region below the first threshold. To obtain an accurate result in the region which is not close to the threshold, it is sufficient to take a small number of terms of the expansion.

In other cases, when one is interested in the region near or beyond the threshold, the expansion for small momenta cannot be used. To obtain expansions for momenta near the threshold and for very large momenta, which may contain square roots and logarithms, a modified procedure would be needed.

In our opinion, momentum expansions could be especially useful in cases where one is interested in a range of values of some of the parameters (e.g. the Higgs and t quark masses in the Standard Model) because the coefficients are obtained analytically. Moreover, the algorithm presented here can also be generalized to calculate massive three-point two-loop diagrams, if all the external momenta squared lie below the corresponding thresholds.

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Fig. 4

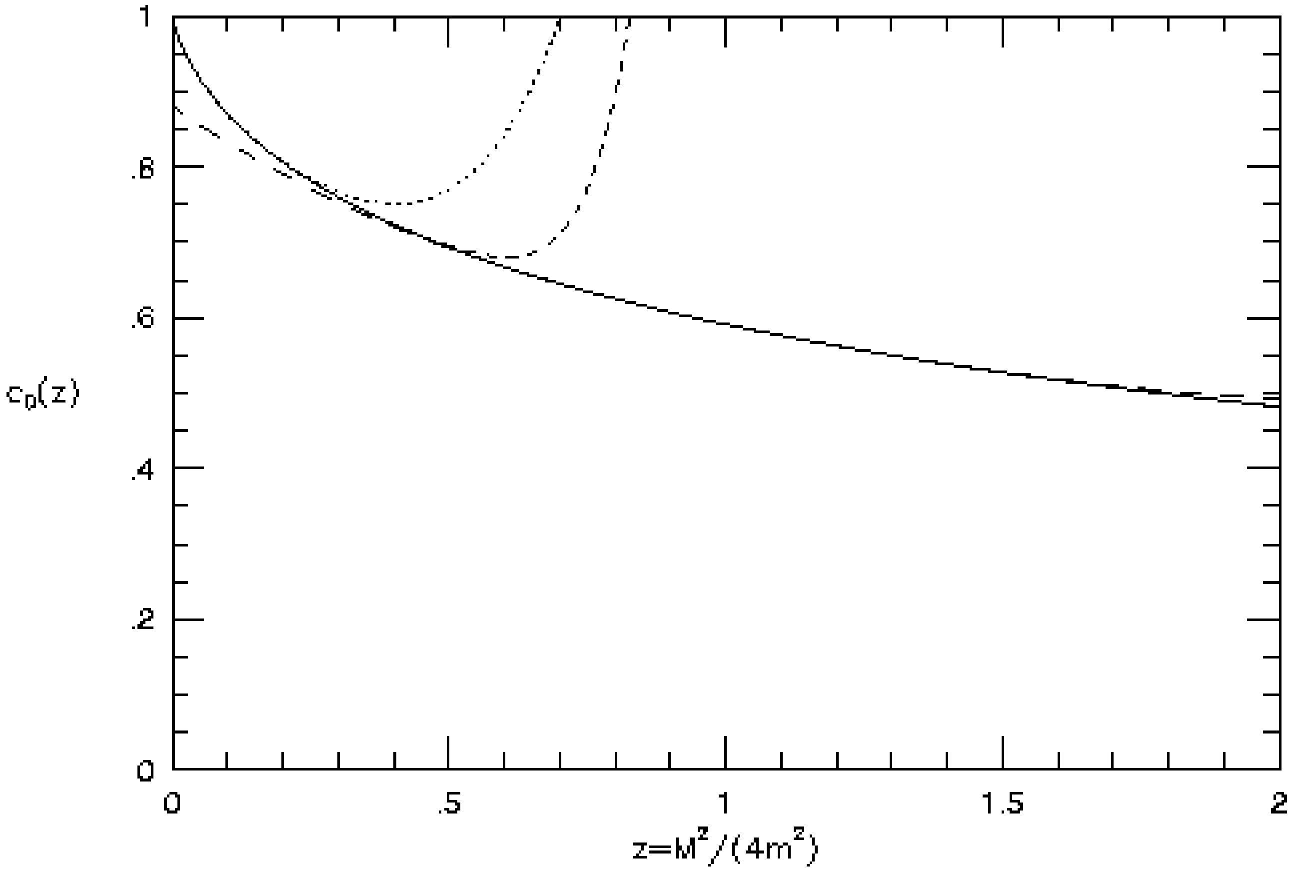


Fig. 5

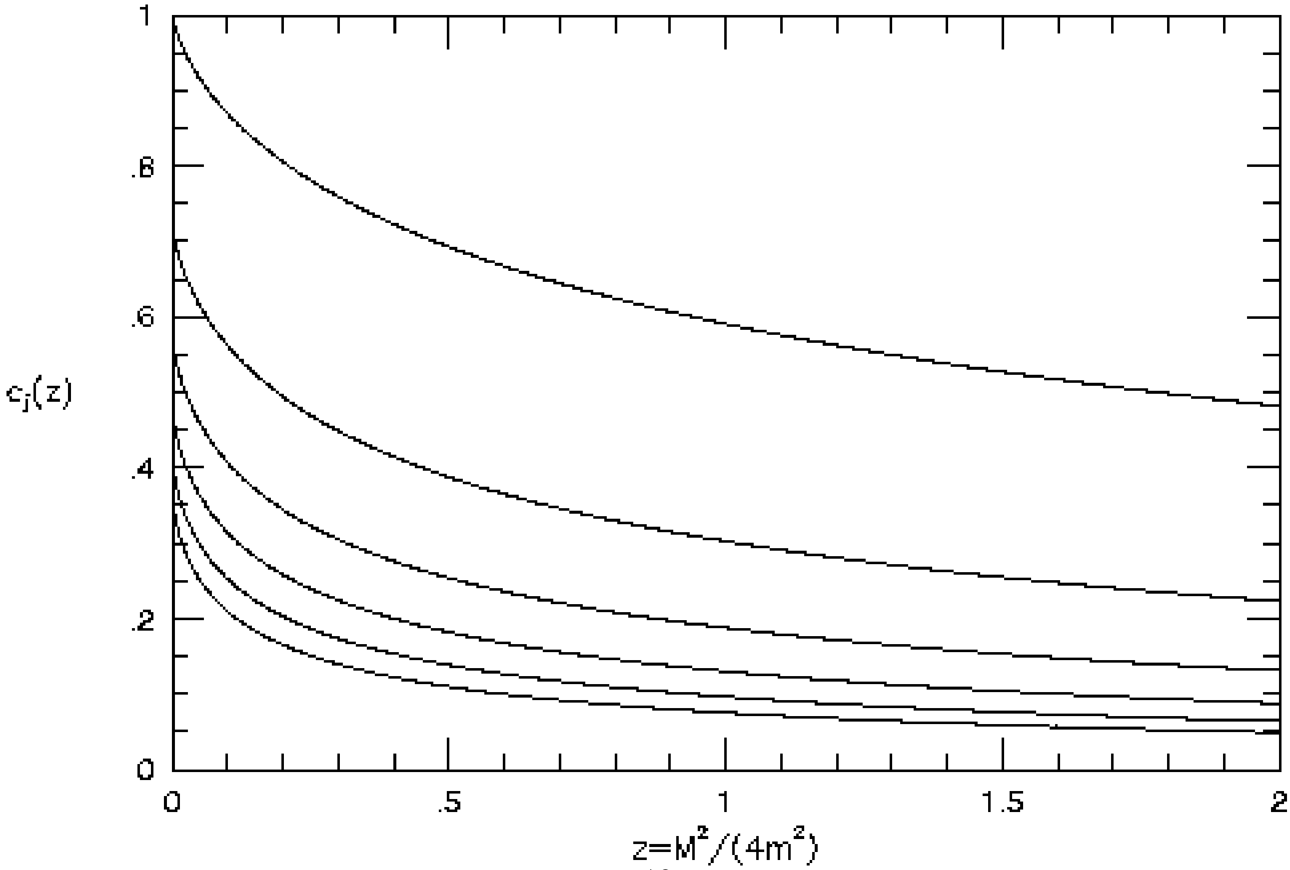
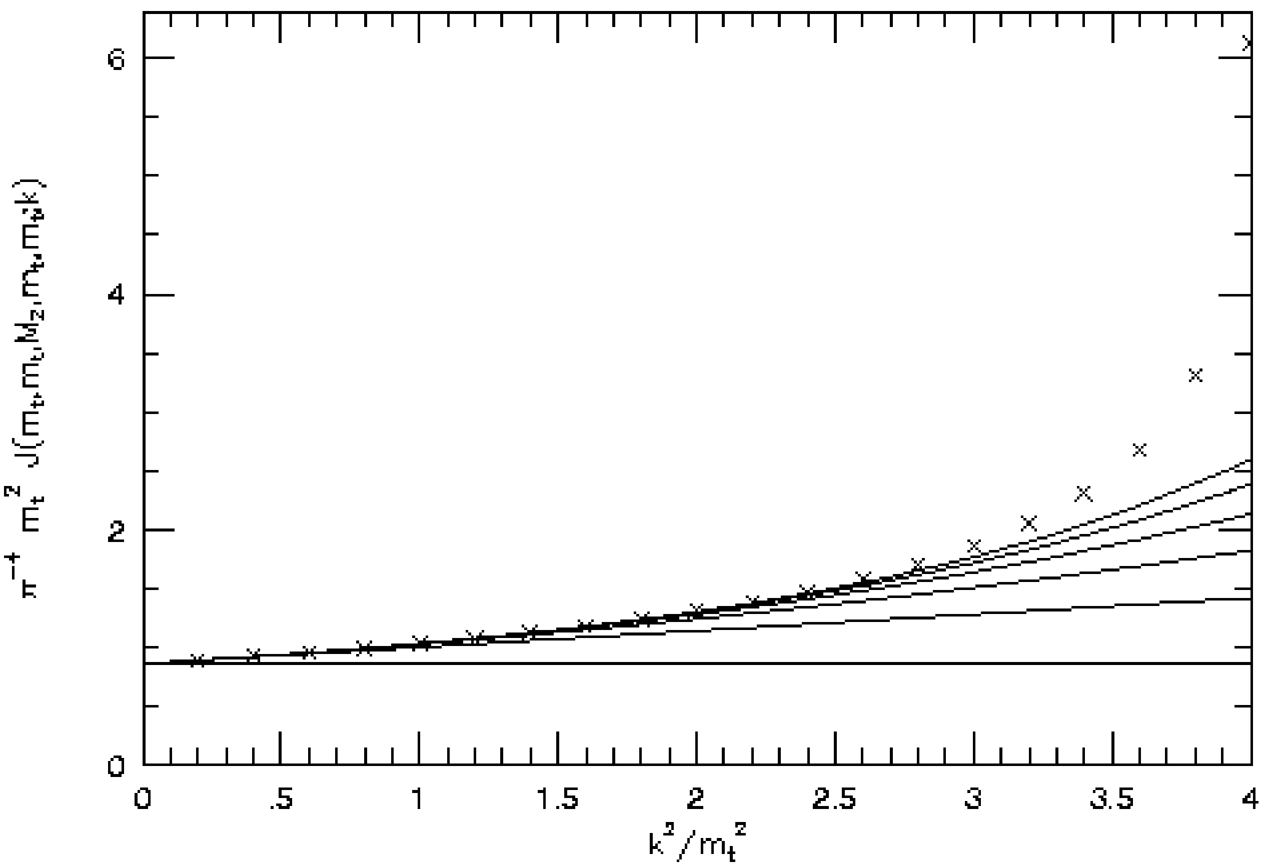
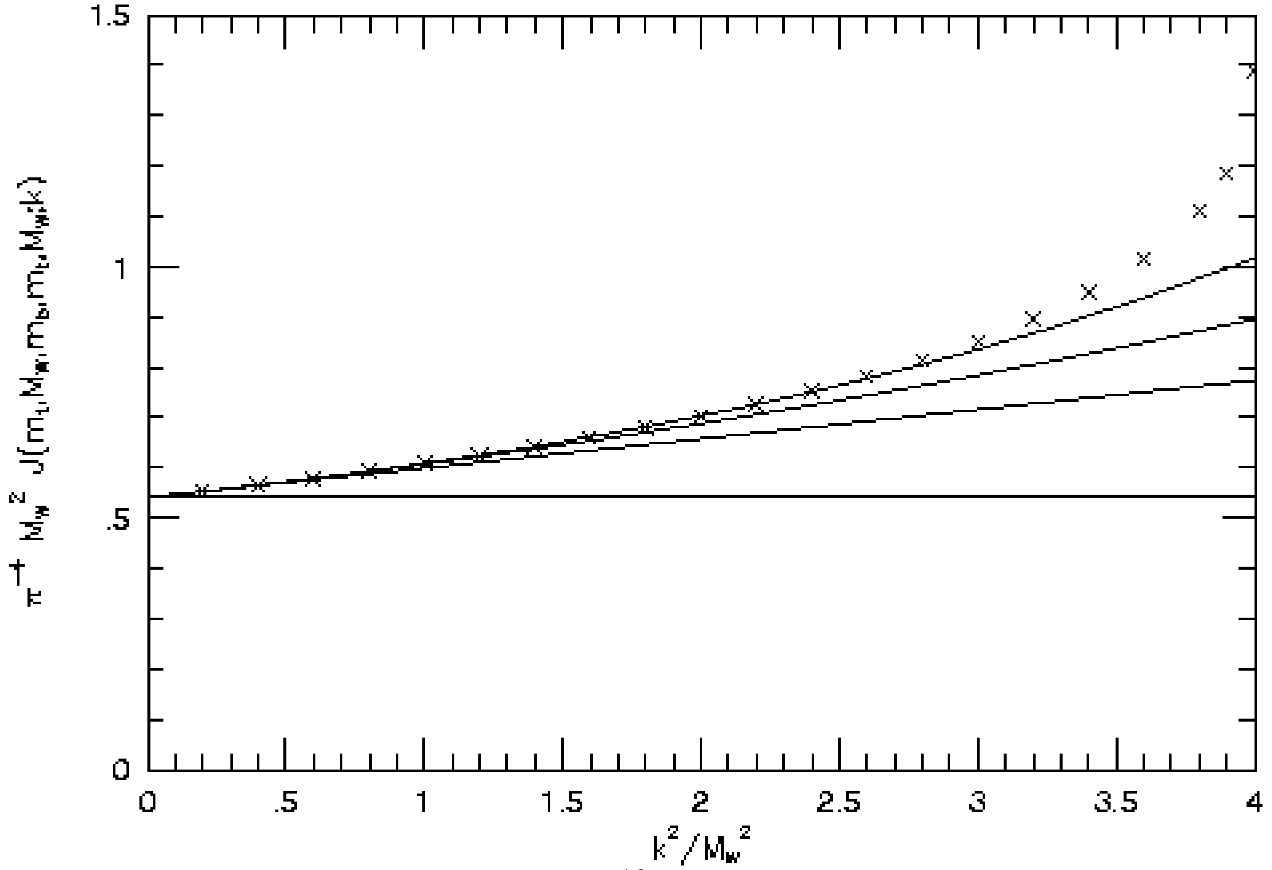


Fig. 6



$\pi^{-4}\ m_t^2\ J(m_t,m_t,M_Z,m_t,m_t;k)$
k^2/m_t^2
0
.5
1
1.5
2
2.5
3
3.5
4
6

Fig. 7



$\pi^{-4}\ M_W^2\ J(m_t,M_W,m_b,m_t,M_W;k)$
k^2/M_W^2
0
.5
1
1.5
2
2.5
3
3.5
4

Fig. 4. The coefficient $c_0(x)$. The solid curve shows the exact value, the curve with long dashes is the expansion near $z = 1$ including terms up to $\mathcal{O}\left((z-1)^4\right)$, the dotted curve is the expansion near $z = 0$ including terms up to $\mathcal{O}\left(z^4 \ln z\right)$, and the curve with short dashes is the expansion near $z = 0$ including terms up to $\mathcal{O}\left(z^{10} \ln z\right)$.

Fig. 5. From top to bottom: $c_0(z)$, $c_1(z)$, $c_2(z)$, $c_3(z)$, $c_4(z)$, $c_5(z)$.

Fig. 6. The integral $J(m_t, m_t, M_Z, m_t, m_t; k)$ with $M_Z = 91.16$ GeV and $m_t = 140$ GeV ($z = 0.106$). The crosses are the values obtained by numerical integration. The straight line is the first term of the momentum expansion. The curves show the improvement of the approximation when successive terms of the momentum expansion are added.

Fig. 7. The same as Fig. 6, but now for $J(m_t, M_W, m_b, m_t, M_W; k)$ with $M_W = 80$ GeV, $m_t = 140$ GeV and $m_b = 5$ GeV.