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## Some exact results for $N$ -point massive Feynman integrals

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### **Abstract**

By using the Mellin–Barnes representation for massive denominators, some exact results for classes of one-loop  $N$ -point massive Feynman integrals are obtained for arbitrary values of the line indices (the powers of denominators) and of the space-time dimension. A representation for corresponding massless integral is also derived.

# 1 Introduction

Much attention has recently been paid to developing methods of evaluation of various types of Feynman diagrams needed for studying many important problems of quantum field theory. In particular, these problems are associated with calculating characteristics of various processes of particles interactions, studying operator expansions, examining behaviours of various Green's functions, etc. The methods, which allow us to evaluate exactly some classes of appropriate Feynman integrals, are of great value. It is often most convenient to use the dimensional regularization [1, 2] (see also the review in Ref. [3]); nevertheless, the analytic (or other) regularization schemes are sometimes useful also.

Up to the present time, the greatest success has been achieved in developing methods of evaluation of massless propagator-type integrals (which depend on one external momentum): the Gegenbauer polynomial technique [4], integration by parts [5], the uniqueness method (see, e.g., Refs. [6, 7]), and some others [8, 9, 10] (see also the review in Ref. [11]). These methods have made it possible to obtain a great number of results for multiloop propagator-type integrals. Massless three-point vertex-type integrals have been studied, e.g., in Refs. [12, 13, 14], while in Ref. [15] four-point diagrams have been considered.

Feynman integrals with massive denominators are known to be more complicated for evaluation. Currently, there are few examples of exact results for classes of massive integrals (some of them have been presented in Refs. [1, 16], see also the references given below). Nevertheless, information about such integrals is often necessary, especially when one deals with heavy particles.

One of the ways of dealing with massive denominators is connected with the application of the  $R^*$  operation [17, 18, 19]. The denominator is expanded in a series with respect to  $m^2/k^2$  ( $m$  is the mass and  $k$  is the momentum of the corresponding line), and the inapplicability of this expansion in the region  $|k^2| < m^2$  is compensated for by appropriate counterterms. Such a way allows one to derive a given finite number of terms of asymptotic expansions of corresponding expressions with respect to the variables of the type of  $m^2/p^2$  ( $p$  is an external momentum).

Some other methods are associated with applying the Mellin transform. In Refs. [20], the massive Feynman amplitudes have been represented in the  $\alpha$ -parametrized form and, afterwards, the Mellin transform has been used to study singularities and asymptotic behaviours of Feynman amplitudes. In Ref. [21] applying the once-through Mellin transform with respect to external squared momentum to evaluation of massive Feynman integrals has been discussed.

In Ref. [22] we have proposed a more straightforward method of dealing with massive denominators. It is based on applying, directly to massive denominators, the Mellin-Barnes representation for the function  ${}_1F_0$ :

$${}_1F_0(\nu|z) \equiv \frac{1}{(1-z)^\nu} = \frac{1}{\Gamma(\nu)} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} ds (-z)^s \Gamma(-s) \Gamma(\nu + s). \quad (1.1)$$

Thus the basic formula of the method can be written as

$$\frac{1}{(k^2 - m^2 + i0)^\nu} = \frac{1}{(k^2 + i0)^\nu} \frac{1}{\Gamma(\nu)} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} ds \left( -\frac{m^2}{k^2 + i0} \right)^s \Gamma(-s) \Gamma(\nu + s), \quad (1.2)$$

where  $\nu$  is the index of the line, and imaginary infinitesimals in denominators define the usual “causal” way of dealing with singularities in the pseudo-Euclidean space. We shall imply below that all squared momenta in denominators have such “additions”, without writing explicitly these infinitesimals. The representation (1.2) is very convenient, since it can be used for the case  $|k^2| > m^2$  as well as for the case  $|k^2| < m^2$ . For example, if  $|k^2| > m^2$  the integrand of (1.2) decreases in the right half-plane of the complex variable  $s$ , and we can evaluate the integral with the help of the residue theorem, closing the contour to the right and summing over the residues at poles of  $\Gamma(-s)$  (in such a way, we obtain an expansion with respect to  $m^2/k^2$ ). On the contrary, if  $|k^2| < m^2$ , we close the contour to the left, and the result can be represented as a sum over the residues of  $\Gamma(\nu + s)$  (in this case we obtain an expansion with respect to the powers of  $k^2/m^2$ ). As is known, turning from closing the contour to the right (in the Mellin–Barnes representation) to closing to the left yields well-known analytic continuation formulae for hypergeometric functions from the variable  $z$  to  $1/z$  [23]. After all massive denominators are represented in the form (1.2), the obtained massless momentum integral must be evaluated and, then, we must represent the remained contour integrals in the form of known functions or expansions (as a rule, the functions of hypergeometric type appear). By using the representation (1.2), in Ref. [22] general results for some classes of one-loop propagator- and vertex-type Feynman integrals have been obtained, which correspond to two- and three-point diagrams.

It should be noted that the appropriateness of applying the Mellin transform and the Mellin–Barnes representation to calculating one-dimensional integrals has been mentioned earlier (see, e. g., Refs. [24, 25]). In particular, in Refs. [26, 13, 27] these methods have been applied to the evaluation of some  $\alpha$ -parametrized integrals (in Ref. [26] a trick analogous to (1.1) has been used when evaluating a parametric integral).

In the present paper we shall use the representation (1.2) to obtain exact results for the classes of the one-loop  $N$ -point massive Feynman integrals that correspond to the “sun”-type diagrams with an arbitrary number of external lines  $N$  (see Fig. 1). We shall consider integrals for arbitrary values of the space-time dimension  $n$  and the indices of the lines. Therefore the results obtained will be useful in both dimensional and analytic regularization schemes. We shall consider only scalar integrals since any integral with a tensor Lorentz structure in the numerator can be reduced to scalar integrals with the help of the substitutions of the type of the formulae listed, e.g., in Ref.[28].

Section 2 illustrates the presented technique through an example of  $N$ -point “vacuum” diagram, in the case when all masses corresponding to internal lines are different. In Section 3 the Mellin–Barnes representation for the  $N$ -point massless diagram with arbitrary external momenta is derived. In Section 4 this expression is used to obtain an exact result for the  $N$ -point massive loop with arbitrary external momenta. The Conclusion (Section 5) formulates and discusses the main results of this paper.

## 2 $N$ -point integrals with zero external momenta and different masses

As an example of applying the presented technique, in this section we shall consider Feynman integrals corresponding to the diagram of Fig. 1, for the case when all ingoing

momenta are zeros ( $p_1 = p_2 = \dots = p_N$ , see Fig. 1). The indices of all lines  $\nu_j$  and corresponding masses  $m_j$  will be considered as arbitrary values. The corresponding integral is of the form

$$I^{(N)}(\{\nu_j\}; \{m_j\}) \equiv \int \frac{d^n k}{(k^2 - m_1^2)^{\nu_1} \dots (k^2 - m_N^2)^{\nu_N}} = \int \frac{d^n k}{\prod_{j=1}^N (k^2 - m_j^2)^{\nu_j}}, \quad (2.1)$$

where  $n = 4 - 2\varepsilon$  is the space-time dimension (in the framework of dimensional regularization [1, 2]). We remember that the ‘‘causal’’ way of dealing with singularities in the pseudo-Euclidean space is implied.

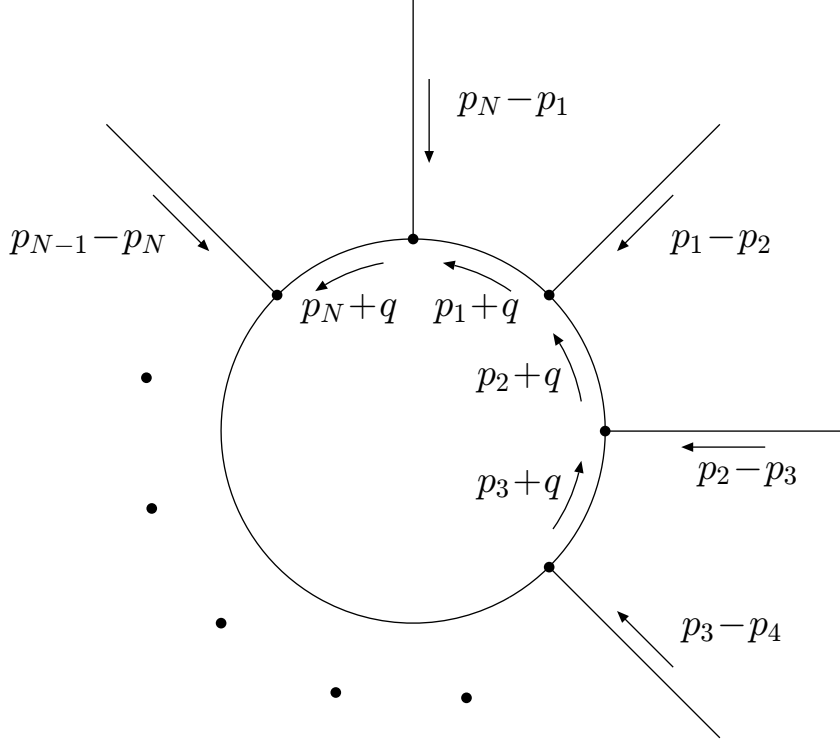


Fig. 1. The arrangement of momenta in the  $N$ -point Feynman diagram

The result for the integral (2.1) at  $N = 1$  is well known [1]:

$$I^{(1)}(\nu; m) = \pi^{n/2} i^{1-n} (-m^2)^{n/2-\nu} \frac{\Gamma(\nu - n/2)}{\Gamma(\nu)}. \quad (2.2)$$

This formula can be easily generalized for the case when we have one massive and one massless denominator:

$$I^{(1)}(\alpha, \nu; 0, m) = \pi^{n/2} i^{1-n} (-m^2)^{n/2-\alpha-\nu} \frac{\Gamma(n/2 - \alpha) \Gamma(\alpha + \nu - n/2)}{\Gamma(\nu) \Gamma(n/2)}. \quad (2.3)$$

Note that the formulae (2.2) and (2.3) can be easily obtained through the technique (1.2), if the property of Ref. [9] is used:

$$I^{(1)}\left(\frac{n}{2} + i\xi; 0\right) = \frac{2\pi^{n/2}}{\Gamma(n/2)} i \pi \delta(\xi).$$

We shall use the well-known results of (2.2) and (2.3) as the basic formulae.

Let us now evaluate the integral (2.1) for arbitrary values of  $N$ . Applying  $N - 1$  times the formula (1.1) to the denominators  $1/(k^2 - m_j^2)^{\nu_j}$  ( $j = l, \dots, N - 1$ ) we obtain the following  $(N - 1)$ -fold integral:

$$I^{(N)}(\{\nu_j\}; \{m_j\}) = \frac{1}{\prod_{j=1}^{N-1} \Gamma(\nu_j)} \frac{1}{(2\pi i)^{N-1}} \int_{-i\infty}^{i\infty} \dots \int_{-i\infty}^{i\infty} \left\{ \prod_{j=1}^{N-1} ds_j (-m_j^2)^{s_j} \Gamma(-s_j) \Gamma(\nu_j + s_j) \right\} \\ \times I^{(2)}\left(\sum_{j=1}^{N-1} (\nu_j + s_j), \nu_N; 0, m_N\right).$$

Inserting the expression (2.3) for  $I^{(2)}$  yields the following representation:

$$I^{(N)}(\{\nu_j\}; \{m_j\}) = \pi^{n/2} i^{1-n} (-m_N^2)^{n/2 - \Sigma \nu_j} \frac{1}{\Gamma(n/2) \Pi \Gamma(\nu_j)} \\ \times \frac{1}{(2\pi i)^{N-1}} \int_{-i\infty}^{i\infty} \dots \int_{-i\infty}^{i\infty} \left\{ \prod_{j=1}^{N-1} ds_j \left(\frac{m_j^2}{m_N^2}\right)^{s_j} \Gamma(-s_j) \Gamma(\nu_j + s_j) \right\} \\ \times \Gamma(n/2 - \Sigma \nu_j + \nu_N - s_1 - \dots - s_{N-1}) \\ \times \Gamma(\Sigma \nu_j - n/2 + s_1 + \dots + s_{N-1}), \quad (2.4)$$

where  $\Sigma$  and  $\Pi$  denote the sum and the product from 1 to  $N$  (in the case when these limits are not written explicitly).

The representation (2.4) is of little use for obtaining a compact result for  $I^{(N)}$ , since we have, with respect to any variable  $s_j$ , two series of poles in the right half-plane as well as in the left half-plane. However, this expression can be transformed by using the formula

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} ds z^s \Gamma(-s) \Gamma(C - s) \Gamma(A + s) \Gamma(B + s) \\ = \frac{\Gamma(A) \Gamma(B) \Gamma(A + C) \Gamma(B + C)}{\Gamma(A + B + C)} {}_2F_1\left(\begin{matrix} A, B \\ A + B + C \end{matrix} \middle| 1 - z\right) \\ = \Gamma(A + C) \Gamma(B + C) \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} ds (z - 1)^s \frac{\Gamma(-s) \Gamma(A + s) \Gamma(B + s)}{\Gamma(A + B + C + s)}, \quad (2.5)$$

where  ${}_2F_1$  is the Gauss' hypergeometric function. The formula (2.5) is a combination of the well-known analytic continuation formula for  ${}_2F_1$  (see, e.g., Ref. [23]) and the Mellin-Barnes representation. Applying the formula (2.5), consecutively, to the integrals with respect to  $s_{N-1}, s_{N-2}, \dots, s_1$  ( $N - 1$  times) we find

$$I^{(N)}(\{\nu_j\}; \{m_j\}) = \pi^{n/2} i^{1-n} (-m_N^2)^{n/2 - \Sigma \nu_j} \frac{1}{\prod_{j=1}^{N-1} \Gamma(\nu_j)} \\ \times \frac{1}{(2\pi i)^{N-1}} \int_{-i\infty}^{i\infty} \dots \int_{-i\infty}^{i\infty} \left\{ \prod_{j=1}^{N-1} ds_j \left(\frac{m_j^2}{m_N^2} - 1\right)^{s_j} \Gamma(-s_j) \Gamma(\nu_j + s_j) \right\} \\ \times \Gamma(\Sigma \nu_j - n/2 + s_1 + \dots + s_{N-1}) [\Gamma(\Sigma \nu_j + s_1 + \dots + s_{N-1})]^{-1}. \quad (2.6)$$

The representation (2.6) is advantageous, in comparison with (2.5), since, for any variable  $s_j$ , it has only one series of the poles of  $\Gamma$  functions in the right half-plane. Therefore, these integrals can be easily evaluated. The result can be represented in the form of the  $(N - 1)$ -fold series of the hypergeometric type:

$$\begin{aligned}
I^{(N)}(\{\nu_j\}; \{m_j\}) &= \pi^{n/2} i^{1-n} (-m_N^2)^{n/2-\Sigma\nu_j} \frac{\Gamma(\Sigma\nu_j - n/2)}{\Gamma(\Sigma\nu_j)} \\
&\times \sum_{j_1=0}^{\infty} \cdots \sum_{j_{N-1}=0}^{\infty} \frac{1}{j_1! \cdots j_{N-1}!} \left(1 - \frac{m_1^2}{m_N^2}\right)^{j_1} \cdots \left(1 - \frac{m_{N-1}^2}{m_N^2}\right)^{j_{N-1}} \\
&\times \frac{(\Sigma\nu_j - n/2)_{j_1+\dots+j_{N-1}} (\nu_1)_{j_1} \cdots (\nu_{N-1})_{j_{N-1}}}{(\Sigma\nu_j)_{j_1+\dots+j_{N-1}}}, \tag{2.7}
\end{aligned}$$

where  $(\nu)_j \equiv \Gamma(\nu + j)/\Gamma(\nu)$  is the Pochhammer symbol. One can see that the series in (2.7) represents the Lauricella hypergeometric function of  $N - 1$  variables,  $F_D^{(N-1)}$  (see the Appendix). The final result is

$$\begin{aligned}
I^{(N)}(\{\nu_j\}; \{m_j\}) &= \pi^{n/2} i^{1-n} (-m_N^2)^{n/2-\Sigma\nu_j} \frac{\Gamma(\Sigma\nu_j - n/2)}{\Gamma(\Sigma\nu_j)} \\
&\times F_D^{(N-1)}\left(\Sigma\nu_j - \frac{n}{2}, \nu_1, \dots, \nu_{N-1}; \Sigma\nu_j \mid 1 - \frac{m_1^2}{m_N^2}, \dots, 1 - \frac{m_{N-1}^2}{m_N^2}\right). \tag{2.8}
\end{aligned}$$

When obtaining series and functions of the hypergeometric type, we shall consider, here and below, that the values of variables correspond to the convergence regions of these series. The expressions in other regions can be obtained either by another choosing dimensionless-making parameters ( $m_N^2$ ) or by using the analytic continuation formulae for hypergeometric functions.

Let us consider some special cases of the formula (2.8).

(a) If some two masses  $m_j$  and  $m_l$  are equal we find, from the properties of the function  $F_D^{(N-1)}$  (see (A.5) and (A.6)), that it reduces to the function  $F_D^{(N-2)}$  which depends on the summary line index  $\nu_j + \nu_l$ . This property is obvious for the integral (2.1), and its fulfillment confirms the correctness of the result (2.8). In particular, if all masses are equal then all arguments of the function  $F_D^{(N-1)}$  (2.8) vanish and we obtain the formula (2.2) (with  $\nu = \Sigma\nu_j$ ).

(b) If one of the masses  $m_1, \dots, m_{N-1}$  vanishes then the corresponding argument is unity, and the function  $F_D^{(N-1)}$  also reduces to the function  $F_D^{(N-2)}$  (see (A.7)). In particular, if some masses are zeros and other masses are equal to each other then we obtain the formula (2.3) where  $\alpha$  is the sum of indices of zero-mass lines and  $\nu$  is the sum of indices of massive lines.

(c) For  $N = 1$ ,  $F_D^{(0)} = 1$ , and we obtain the formula (2.2). For  $N = 2$ ,  $F_D^{(1)}$  corresponds to  ${}_2F_1$ , and we have

$$\begin{aligned}
I^{(2)}(\nu_1, \nu_2; m_1, m_2) &= \pi^{n/2} i^{1-n} (-m_2^2)^{n/2-\nu_1-\nu_2} \\
&\times \frac{\Gamma(\nu_1 + \nu_2 - n/2)}{\Gamma(\nu_1 + \nu_2)} {}_2F_1\left(\begin{matrix} \nu_1 + \nu_2 - n/2, \nu_1 \\ \nu_1 + \nu_2 \end{matrix} \mid 1 - \frac{m_1^2}{m_2^2}\right). \tag{2.9}
\end{aligned}$$

If the function  ${}_2F_1$  in (2.9) is continued analytically to the variable  $m_1^2/m_2^2$  then a special case of the formula for the propagator-type integrals (see Ref. [22]), when the external momentum vanishes, could be obtained. In particular, if  $\nu_1 = \nu_2 = 1$  then we have the known formula (see Ref. [18])

$$I^{(2)}(1, 1; m_1, m_2) = -i\pi^{n/2} \Gamma(1 - n/2) \frac{(m_2^2)^{n/2-1} - (m_1^2)^{n/2-1}}{m_2^2 - m_1^2}.$$

It should be also noted that, for  $N = 3$ ,  $F_D^{(2)}$  corresponds to the Appell's hypergeometric function of two variables  $F_1$  (see Ref. [29] and (A.3)).

### 3 Representation for $N$ -point massless integrals

Let us now consider integrals corresponding to the diagram of Fig. 1, with arbitrary external momenta. It is clear that applying the technique (1.2) to massive loops of the type of Fig. 1 will require information about appropriate massless integrals. In this section, we shall consider such massless  $N$ -point integrals that are of the form

$$J^{(N)}(\{\nu_j\}|\{p_j\}; 0) \equiv \int \frac{d^n r}{[(p_1 + r)^2]^{\nu_1} \dots [(p_N + r)^2]^{\nu_N}} = \int \frac{d^n r}{\prod_{j=1}^N [(p_j + r)^2]^{\nu_j}}. \quad (3.1)$$

Using the standard  $\alpha$ -representation technique and integrating over the momentum  $r$ , we find

$$J^{(N)}(\{\nu_j\}|\{p_j\}; 0) = \frac{i^{1-n/2-\Sigma\nu_j} \pi^{n/2}}{\prod \Gamma(\nu_j)} \int_0^\infty \dots \int_0^\infty \frac{\prod \alpha_j^{\nu_j-1} d\alpha_j}{(\Sigma \alpha_j)^{n/2}} \exp\left\{i \left[ \Sigma \alpha_j p_j^2 - \frac{(\Sigma \alpha_j p_j)^2}{\Sigma \alpha_j} \right]\right\}.$$

Performing the standard change of the integration variables ( $\Sigma \alpha_j = \lambda$ ,  $\alpha_j = \lambda \beta_j$ ;  $\Sigma \beta_j = 1$ ) and integrating over  $\lambda$ , we have

$$J^{(N)}(\{\nu_j\}|\{p_j\}; 0) = \pi^{n/2} i^{1-n} \frac{\Gamma(\Sigma \nu_j - n/2)}{\prod \Gamma(\nu_j)} \int_0^1 \dots \int_0^1 \frac{\prod \beta_j^{\nu_j-1} d\beta_j \delta(\Sigma \beta_j - 1)}{\left\{ \sum_{j<l} \beta_j \beta_l k_{jl}^2 \right\}^{\Sigma \nu_j - n/2}}, \quad (3.2)$$

where  $k_{jl} \equiv p_j - p_l$ ,  $j < l$ . The total number of independent invariants  $k_{jl}^2$  is  $L \equiv N(N-1)/2$ . We have, among these invariants,  $N$  external momenta squared (see Fig. 1):  $k_{12}^2, k_{23}^2, \dots, k_{N-1,N}^2, k_{1N}^2$ . Note that at  $N = 2$  we obtain, from (3.2), the well-known result

$$J^{(2)}(\nu_1, \nu_2 | p_1, p_2; 0) = \pi^{n/2} i^{1-n} [(p_1 - p_2)^2]^{n/2-\nu_1-\nu_2} \frac{\Gamma(n/2-\nu_1) \Gamma(n/2-\nu_2) \Gamma(\nu_1+\nu_2-n/2)}{\Gamma(\nu_1) \Gamma(\nu_2) \Gamma(n-\nu_1-\nu_2)}. \quad (3.3)$$

In the general case, the denominator of the integrand of (3.2) represents a sum of  $L = N(N-1)/2$  items, which is raised to the power  $(\Sigma \nu_j - n/2)$ . It is convenient, for dealing with this denominator, to use the  $(L-1)$ -fold Mellin-Barnes representation:

$$\frac{1}{\left(\sum_{j=1}^L z_j\right)^a} = \frac{1}{\Gamma(a)} \frac{1}{(z_L)^a} \frac{1}{(2\pi i)^{L-1}} \int_{-\infty}^{i\infty} \dots \int_{-\infty}^{i\infty} \left\{ \prod_{j=1}^{L-1} dt_j \left(\frac{z_j}{z_L}\right)^{t_j} \Gamma(-t_j) \right\} \Gamma\left(a + \sum_{j=1}^{L-1} t_j\right). \quad (3.4)$$

The formula (3.4) can be obtained by applying,  $L - 1$  times, the formula (1.1) to the functions of the type of  ${}_1F_0$ . Note that, for the case  $N = 3$ , the formula analogous to (3.4) (with  $a = 1$ ) has been used, e.g., in Ref. [27].

After the denominator in (3.2) is represented in the form (3.4), we can integrate over all  $\beta$ 's by using the known formula

$$\int_0^1 \dots \int_0^1 \prod \beta_i^{\rho_i-1} d\beta_i \delta(\sum \beta_i - 1) = \frac{\prod \Gamma(\rho_i)}{\Gamma(\sum \rho_i)}$$

where, in our case,  $\rho_i = \nu_i + \sigma_i$  with

$$\sigma_i \equiv \sum_{j<i} s_{ji} + \sum_{l>i} s_{il}; \quad \sum_{i=1}^N \sigma_i = 2 \sum_{j<l} s_{jl}. \quad (3.5)$$

As a result, the following symmetric representation of the integral (3.1) can be obtained:

$$\begin{aligned} J^{(N)}(\{\nu_j\}|\{p_j\}; 0) &= \pi^{n/2} i^{1-n} \frac{1}{\Gamma(n - \sum \nu_i) \prod \Gamma(\nu_i)} \\ &\times \frac{1}{(2\pi i)^{L-1}} \int_{-i\infty}^{i\infty} \dots \int_{-i\infty}^{i\infty} \Delta\left(\sum \nu_i - \frac{n}{2} + \sum_{j<l} s_{jl}\right) \prod'_{j<l} ds_{jl} \\ &\times \prod_{j<l} \left\{ (k_{jl}^2)^{s_{jl}} \Gamma(-s_{jl}) \right\} \prod_{i=1}^N \Gamma\left(\nu_i + \sum_{j<i} s_{ji} + \sum_{l>i} s_{il}\right), \quad (3.6) \end{aligned}$$

where  $\Delta\left(\sum \nu_i - \frac{n}{2} + \sum_{j<l} s_{jl}\right)$  shows that we are on the surface  $\left\{\sum \nu_i - \frac{n}{2} + \sum_{j<l} s_{jl} = 0\right\}$  and, therefore, one of  $s_{jl}$  should be expressed through all other variables;  $\prod'_{j<l} ds_{jl}$  denotes the product of all  $ds_{jl}$  except one chosen variable. Let us express, for example,  $s_{1N}$  through all other variables:

$$s_{1N} = \frac{n}{2} - \sum \nu_i - \sum_{\substack{j<l \\ (j,l) \neq (1,N)}} s_{jl}. \quad (3.7)$$

Then the symmetry in (3.6) is broken, and we obtain, finally, the following  $(L - 1)$ -fold Mellin-Barnes representation of the massless integral (3.1) ( $L \equiv N(N - 1)/2$ ):

$$\begin{aligned} J^{(N)}(\{\nu_j\}|\{p_j\}; 0) &= \pi^{n/2} i^{1-n} (k_{1N}^2)^{n/2 - \sum \nu_i} \frac{1}{\Gamma(n - \sum \nu_i) \prod \Gamma(\nu_i)} \\ &\times \frac{1}{(2\pi i)^{L-1}} \int_{-i\infty}^{i\infty} \dots \int_{-i\infty}^{i\infty} \prod_{\substack{j<l \\ (j,l) \neq (1,N)}} \left\{ ds_{jl} \left( \frac{k_{jl}^2}{k_{1N}^2} \right)^{s_{jl}} \Gamma(-s_{jl}) \right\} \\ &\times \Gamma\left(\sum \nu_i - \frac{n}{2} + \sum_{\substack{j<l \\ (j,l) \neq (1,N)}} s_{jl}\right) \prod_{i=2}^{N-1} \Gamma\left(\nu_i + \sum_{j<i} s_{ji} + \sum_{l>i} s_{il}\right) \\ &\times \Gamma\left(\frac{n}{2} - \sum \nu_i + \nu_1 - \sum_{\substack{j<l \\ j \neq 1}} s_{jl}\right) \Gamma\left(\frac{n}{2} - \sum \nu_i + \nu_N - \sum_{\substack{j<l \\ j \neq N}} s_{jl}\right). \quad (3.8) \end{aligned}$$

For  $N = 3$  we obtain here the known result [13] (see also Ref. [22]) which can be expressed in terms of the Appell's hypergeometric functions of two variables  $F_4$  (or in terms of the Mejer's  $G$ -function of two variables, see Ref. [30]). For  $N \geq 4$ , it is considerably more complicated to express the result in terms of the known hypergeometric functions since for any  $s_{jl}$  we have several series of poles of  $\Gamma$  functions in both right and left half-planes (by analogy with (2.4)). However, the following section will show that it is sufficient to have the representation (3.8) to obtain results for corresponding massive integrals.

## 4 $N$ -point integrals with equal masses

Let us consider the class of  $N$ -point Feynman integrals corresponding to the diagram of Fig. 1, for which all masses corresponding to the internal segments of the loop are of the same value  $m$  ( $m_1 = \dots = m_N \equiv m$ ) and all indices of the lines are arbitrary:

$$J^{(N)}(\{\nu_j\}|\{p_j\}; m) \equiv \int \frac{d^n r}{[(p_1+r)^2 - m^2]^{\nu_1} \dots [(p_N+r)^2 - m^2]^{\nu_N}} = \int \frac{d^n r}{\prod_{j=1}^N [(p_j+r)^2 - m^2]^{\nu_j}}. \quad (4.1)$$

Applying the basic formula of the method (1.2) to each of the  $N$  denominators yields

$$J^{(N)}(\{\nu_j\}|\{p_j\}; m) = \frac{1}{\prod \Gamma(\nu_i)} \frac{1}{(2\pi i)^N} \int_{-i\infty}^{i\infty} \dots \int_{-i\infty}^{i\infty} \left\{ \prod_{i=1}^N dt_i \Gamma(-t_i) \Gamma(\nu_i + t_i) \right\} \times (-m^2)^{\sum t_i} J^{(N)}(\{\nu_j + t_j\}|\{p_j\}; 0), \quad (4.2)$$

where the zero-mass integral is defined by the formula (3.1). Inserting the representation (3.8) (obtained in the previous section) into (4.2) we get the  $(N+L-1)$ -fold Mellin-Barnes integral ( $L = N(N-1)/2$ ).

We shall show that, by using the Barnes lemma, one can take integrals over all  $t_i$  (4.2) except one variable. The Barnes lemma is of the following form (see, e.g., Ref. [24]):

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} ds \Gamma(A+s) \Gamma(B+s) \Gamma(C-s) \Gamma(D-s) = \frac{\Gamma(A+C) \Gamma(B+C) \Gamma(A+D) \Gamma(B+D)}{\Gamma(A+B+C+D)}. \quad (4.3)$$

For example, consider an integral with respect to the pair of variables,  $t_a$  and  $t_b$  (such that  $(a, b) \neq (1, N)$ ). The power of the variable in the integrand depends only on the sum,  $t_{ab} = t_a + t_b$ :  $(-m^2/k_{1N}^2)^{t_a+t_b}$ . Putting  $t_b = t_{ab} - t_a$  and using (4.3) one can integrate with respect to  $t_a$ :

$$\begin{aligned} & \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dt_a \Gamma(-t_a) \Gamma(-t_{ab} + t_a) \Gamma(\nu_a + \sigma_a + t_a) \Gamma(\nu_b + \sigma_b + t_{ab} - t_a) \\ &= \frac{\Gamma(\nu_a + \sigma_a) \Gamma(\nu_b + \sigma_b)}{\Gamma(\nu_a + \nu_b + \sigma_a + \sigma_b)} \Gamma(-t_{ab}) \Gamma(\nu_a + \nu_b + \sigma_a + \sigma_b + t_{ab}), \end{aligned}$$

where  $\sigma_i$  are defined by the formula (3.5).

Applying, consecutively,  $N - 1$  times the Barnes lemma (4.3) (including the variables  $t_1$  and  $t_N$ ) and taking into account that

$$\sum_{i=2}^{N-1} \sigma_i = \sum_{\substack{j<l \\ j \neq 1}} s_{jl} + \sum_{\substack{j<l \\ j \neq N}} s_{jl} ,$$

we find

$$\begin{aligned} J^{(N)}(\{\nu_j\}|\{p_j\}; m) &= \pi^{n/2} i^{1-n} (k_{1N}^2)^{n/2-\Sigma\nu_i} \frac{1}{\prod\Gamma(\nu_i)} \\ &\times \frac{1}{(2\pi i)^{L-1}} \int_{-i\infty}^{i\infty} \dots \int_{-i\infty}^{i\infty} \prod_{\substack{j<l \\ (j,l) \neq (1,N)}} \left\{ ds_{jl} \left( \frac{k_{jl}^2}{k_{1N}^2} \right)^{s_{jl}} \Gamma(-s_{jl}) \right\} \\ &\times \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dt \left( -\frac{m^2}{k_{1N}^2} \right)^t \frac{\Gamma(-t)}{\Gamma(n - \Sigma\nu_i - 2t)} \\ &\times \Gamma\left( \sum \nu_i - \frac{n}{2} + \sum_{\substack{j<l \\ (j,l) \neq (1,N)}} s_{jl} + t \right) \prod_{i=2}^{N-1} \Gamma\left( \nu_i + \sum_{j<i} s_{ji} + \sum_{l>i} s_{il} \right) \\ &\times \Gamma\left( \frac{n}{2} - \sum \nu_i + \nu_1 - \sum_{\substack{j<l \\ j \neq 1}} s_{jl} - t \right) \\ &\times \Gamma\left( \frac{n}{2} - \sum \nu_i + \nu_N - \sum_{\substack{j<l \\ j \neq N}} s_{jl} - t \right) , \end{aligned} \quad (4.4)$$

where  $t \equiv \Sigma t_i$ . Restoring, finally, the variable  $s_{1N}$  with the help of the substitution

$$t = \frac{n}{2} - \sum \nu_i - \sum_{\substack{j<l \\ (j,l) \neq (1,N)}} s_{jl} - s_{1N} ,$$

we obtain the following symmetric  $L$ -fold Mellin–Barnes representation of the integral (4.1):

$$\begin{aligned} J^{(N)}(\{\nu_j\}|\{p_j\}; m) &= \pi^{n/2} i^{1-n} (-m^2)^{n/2-\Sigma\nu_i} \frac{1}{\prod\Gamma(\nu_i)} \\ &\times \frac{1}{(2\pi i)^L} \int_{-i\infty}^{i\infty} \dots \int_{-i\infty}^{i\infty} \prod_{j<l} \left\{ ds_{jl} \left( -\frac{k_{jl}^2}{m^2} \right)^{s_{jl}} \Gamma(-s_{jl}) \right\} \\ &\times \Gamma\left( \sum \nu_i - \frac{n}{2} + \sum_{j<l} s_{jl} \right) \left[ \Gamma\left( \sum \nu_i + 2 \sum_{j<l} s_{jl} \right) \right]^{-1} \\ &\times \prod_{i=1}^N \Gamma\left( \nu_i + \sum_{j<i} s_{ji} + \sum_{l>i} s_{il} \right) . \end{aligned} \quad (4.5)$$

Note that the ‘‘symmetry breakdown’’ (as in (3.8)) is not required here, since we have an ‘‘external’’ dimensionless-making parameter  $m^2$  (in this sense, the massive integrals

appear to be simpler than the massless ones). As in the case of (2.6), we have here only one series of poles of  $\Gamma$  functions in the integrand in the right half-plane of any variable  $s_{jl}$  ( $\Gamma(-s_{jl})$ ). Therefore, we can easily obtain a representation in the form of the following  $L$ -fold hypergeometric series:

$$J^{(N)}(\{\nu_j\}|\{p_j\}; m) = \pi^{n/2} i^{1-n} (-m^2)^{n/2-\Sigma\nu_i} \frac{\Gamma(\Sigma\nu_i-n/2)}{\Gamma(\Sigma\nu_i)} \sum_{\{a_{jl}: j<l\}} \cdots \sum_{\{a_{jl}: j<l\}} \left\{ \prod_{j<l} \frac{1}{a_{jl}!} \left( \frac{k_{jl}^2}{4m^2} \right)^{a_{jl}} \right\} \\ \times \frac{(\Sigma\nu_i-n/2)_{\Sigma a_{jl}}}{(\Sigma\nu_i/2)_{\Sigma a_{jl}} ((\Sigma\nu_i+1)/2)_{\Sigma a_{jl}}} \prod_{i=1}^N (\nu_i)_{\Sigma_{j<i} a_{ji} + \Sigma_{l>i} a_{il}}, \quad (4.6)$$

where  $\Sigma a_{jl} \equiv \sum_{j<l} \sum a_{jl}$  and  $(\nu)_a \equiv \Gamma(\nu+a)/\Gamma(\nu)$  is the Pochhammer symbol. As before, we suppose that the values of variables correspond to the convergence regions of occurring hypergeometric series. When obtaining the formula (4.6), we have used the well-known gamma-function duplication formula (see, e.g., Ref. [23]). It follows from this formula that

$$(\Sigma\nu_i)_{2\Sigma a_{jl}} = (\Sigma\nu_i/2)_{\Sigma a_{jl}} ((\Sigma\nu_i+1)/2)_{\Sigma a_{jl}} 4^{\Sigma a_{jl}}.$$

The formula (4.6) can be also represented in the form of the generalized Lauricella hypergeometric function of  $L$  variables [31] (see the Appendix):

$$J^{(N)}(\{\nu_j\}|\{p_j\}; m) = \pi^{n/2} i^{1-n} (-m^2)^{n/2-\Sigma\nu_i} \frac{\Gamma(\Sigma\nu_i-n/2)}{\Gamma(\Sigma\nu_i)} \\ \times F_{2:0;\dots;0}^{N+1:0;\dots;0} \left[ \begin{matrix} (\Sigma\nu_i-n/2 : L \text{ units}), \{(\nu_i : N-1 \text{ units}, L-N+1 \text{ zeros})\} \\ (\Sigma\nu_i/2 : L \text{ units}), ((\Sigma\nu_i+1)/2 : L \text{ units}) \end{matrix} \middle| \left\{ \frac{k_{jl}^2}{4m^2} \right\} \right], \quad (4.7)$$

where  $L = N(N-1)/2$  is the number of arguments of the function (the number of independent  $k_{jl}^2$ ). The order of the arrangement of zeros and units at the parameters  $\nu_i$  depends on the order of the arrangement of the arguments and follows from the form of the expression (4.6).

The representation in terms of hypergeometric functions (4.7) (or in the form of the Mellin–Barnes integral (4.5)) allows one, in some cases, to continue analytically the hypergeometric series (4.6) to other variables (e.g.,  $m^2/k_{jl}^2$ , etc.). In the general case, however, the analytic continuation formulae of the generalized Lauricella hypergeometric function have not been sufficiently examined.

Consider some special cases of the obtained exact results (4.6) and (4.7).

(a) Suppose, some ingoing momentum (see Fig. 1) is zero (e.g.,  $k_{N-1,N} = 0$ ). In this case, for all  $j \leq N-2$  we have  $k_{jN} = k_{j,N-1}$  and we obtain that one of the arguments ( $k_{N-1,N}^2$ ) vanishes and  $N-2$  pairs of arguments coincide ( $k_{jN}^2 = k_{j,N-1}^2$ ,  $j \leq N-2$ ). Therefore, the number of independent arguments,  $L'$ , turns out to be

$$L' = N(N-1)/2 - (N-2) - 1 = (N-1)(N-2)/2,$$

i.e., it corresponds to the number of external lines equal to  $N-1$ . In addition, one can easily verify, from the explicit form of (4.6), that in this case a hypergeometric series of the same type can be obtained, which corresponds to the  $(N-1)$ -point diagram,  $(N-1)$ th index of the line being equal to the sum  $(\nu_{N-1} + \nu_N)$ . This property is obvious for the

considered integrals (4.1), and it confirms the correctness of the expressions (4.6) and (4.7) obtained. In particular, if all ingoing momenta are zeros, all  $k_{jl}$  vanish, and we get the well-known result (2.2) [1] with  $\nu = \Sigma \nu_j$ .

(b) For  $N = 1$ , the sum in (4.6) should be taken as unity, and we also come to the same result (2.2). For  $N = 2$  we find, from (4.7), that

$$J^{(2)}(\nu_1, \nu_2 | p_1, p_2; m) = \pi^{n/2} i^{1-n} (-m^2)^{n/2-\nu_1-\nu_2} \frac{\Gamma(\nu_1 + \nu_2 - n/2)}{\Gamma(\nu_1 + \nu_2)} \times {}_3F_2 \left( \begin{matrix} \nu_1, \nu_2, \nu_1 + \nu_2 - n/2 \\ (\nu_1 + \nu_2)/2, (\nu_1 + \nu_2 + 1)/2 \end{matrix} \middle| \frac{k_{12}^2}{4m^2} \right), \quad (4.8)$$

where  $k_{12} \equiv p_1 - p_2$ . This expression coincides with the result obtained in Ref. [22]. The formula (4.8) can be easily analytically continued to the variable  $4m^2/k_{12}^2$  (see Ref. [22]). Note that the result for the three-point integral also coincides with the expression obtained in Ref. [22].

## 5 Conclusion

In the present paper, by using the technique of the Mellin–Barnes representation for massive denominators (1.2), we have obtained exact expressions for the classes of Feynman integrals, (2.1) and (4.1), which correspond to one-loop  $N$ -point diagrams with massive denominators. In addition, the Mellin–Barnes representation for corresponding massless  $N$ -point integral (3.1) has been obtained. All results have been obtained for the case when the indices of the lines  $\nu_j$  and the space-time dimension  $n$  are arbitrary. This fact allows for using these results in the framework of both dimensional and analytic regularization schemes. The results for massive integrals are presented in the form of hypergeometric functions. This gives us the possibility, when appropriate analytic continuation formulae are known, to turn to other variables and to study various regions of momenta (see, e.g., Ref. [22]). Although we have evaluated integrals in the pseudo-Euclidean momentum space, turning to Euclidean expressions can be performed without difficulty. It should be noted that the obtained formulae are considerably simplified if we consider the cases when some squared momenta vanish or are equal to the threshold values. The expansion in  $\varepsilon = (4 - n)/2$  also simplifies these expressions. For two- and three-point diagrams, such cases have been examined in Ref. [22].

The obtained results enlarge a number of classes of exactly evaluated Feynman diagrams in quantum field theory. These results can be applied, for example, to cases when the one-loop contributions to processes  $\{ M \text{ particles} \rightarrow (N - M) \text{ particles} \}$  ( $M < N$ ) are considered. One can use these expressions and representations as the “blocks” in appropriate multiloop calculations. Exact results allow us to study the behaviours of diagrams for various respective values of the masses of “external” and “internal” particles. This can be useful in an investigation of theories where some masses of the particles are unknown and their values are not fixed (e.g., the Higgs sector of the electroweak model, etc.). In addition, these results can be applied to the examination of the effective potential, fermionic determinants, threshold effects, etc.

The present paper demonstrates the convenience of applying the technique (1.2) to evaluating massive Feynman integrals. This method can be also used in evaluation of

multiloop integrals with massive denominators, axial-gauge massive integrals, etc. A subsequent development of this method can enlarge a number of exact results for massive Feynman diagrams.

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## A Hypergeometric functions occurring in the paper

In this appendix we present definitions and some properties of hypergeometric functions occurring in the present paper. We suppose that the values of variables correspond to the convergence regions of corresponding hypergeometric series. The expansions in other regions of the variables can be obtained by using the analytic continuation formulae for hypergeometric functions. These and other properties of functions considered can be found, e.g., in Refs. [23, 29, 31, 32, 33, 34].

The generalized hypergeometric function of one variable is defined by

$${}_A F_C \left( \begin{matrix} a_1, \dots, a_A \\ c_1, \dots, c_C \end{matrix} \middle| z \right) = \sum_{j=0}^{\infty} \frac{(a_1)_j \dots (a_A)_j}{(c_1)_j \dots (c_C)_j} \frac{z^j}{j!}, \quad (\text{A.1})$$

where  $(a)_j \equiv \Gamma(a+j)/\Gamma(a)$  is the Pochhammer symbol. In particular, for  $A = 2$  and  $B = 1$  we obtain here the Gauss' hypergeometric function,  ${}_2F_1$ ; and for  $A = 1$  and  $B = 0$  we have the function  ${}_1F_0(a|z) = (1-z)^{-a}$  (see (1.1)). The value of the function  ${}_2F_1$  with unit argument is

$${}_2F_1 \left( \begin{matrix} a, b \\ c \end{matrix} \middle| 1 \right) = \frac{\Gamma(c) \Gamma(c-a-b)}{\Gamma(c-a) \Gamma(c-b)}. \quad (\text{A.2})$$

The Appell's hypergeometric function of two variables  $F_1$  is of the following form:

$$F_1(a, b_1, b_2; c | z_1, z_2) = \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \frac{(a)_{j_1+j_2} (b_1)_{j_1} (b_2)_{j_2}}{(c)_{j_1+j_2}} \frac{z_1^{j_1} z_2^{j_2}}{j_1! j_2!}. \quad (\text{A.3})$$

The Lauricella hypergeometric function of  $L$  variables,  $F_D^{(L)}$ , is defined by

$$F_D^{(L)}(a, b_1, \dots, b_L; c | z_1, \dots, z_L) = \sum_{j_1=0}^{\infty} \dots \sum_{j_L=0}^{\infty} \frac{(a)_{j_1+\dots+j_L} (b_1)_{j_1} \dots (b_L)_{j_L}}{(c)_{j_1+\dots+j_L}} \frac{z_1^{j_1} \dots z_L^{j_L}}{j_1! \dots j_L!}. \quad (\text{A.4})$$

One can easily see that, e.g.,  $F_D^{(2)} = F_1$ . The following reduction formulae follow from the definition (A.4):

$$F_D^{(L)}(a, b_1, \dots, b_{L-1}, b_L; c | z_1, \dots, z_{L-1}, 0) = F_D^{(L-1)}(a, b_1, \dots, b_{L-1}; c | z_1, \dots, z_{L-1}), \quad (\text{A.5})$$

$$F_D^{(L)}(a, b_1, b_2, \dots, b_L; c | z, z, z_3, \dots, z_L) = F_D^{(L-1)}(a, b_1 + b_2, b_3, \dots, b_L; c | z, z_3, \dots, z_L), \quad (\text{A.6})$$

$$F_D^{(L)}(a, b_1, \dots, b_{L-1}, b_L; c | z_1, \dots, z_{L-1}, 1) = \frac{\Gamma(c) \Gamma(c - a - b_L)}{\Gamma(c - a) \Gamma(c - b_L)} \\ \times F_D^{(L-1)}(a, b_1, \dots, b_{L-1}; c - b_L | z_1, \dots, z_{L-1}). \quad (\text{A.7})$$

When obtaining the latter formula, it is convenient to use the property (A.2).

The generalized Lauricella function of  $L$  variables has been introduced in Ref. [31]. It is of the following form:

$$F_{C:D^{(1)};\dots;D^{(L)}}^{A:B^{(1)};\dots;B^{(L)}} \left[ \begin{array}{l} [a : \alpha^{(1)}, \dots, \alpha^{(L)}] : [b^{(1)} : \beta^{(1)}]; \dots; [b^{(L)} : \beta^{(L)}] \\ [c : \gamma^{(1)}, \dots, \gamma^{(L)}] : [d^{(1)} : \delta^{(1)}]; \dots; [d^{(L)} : \delta^{(L)}] \end{array} \middle| z_1, \dots, z_L \right] \\ = \sum_{j_1=0}^{\infty} \dots \sum_{j_L=0}^{\infty} \frac{\prod_{i=1}^A (a_i)_{\alpha_i^{(1)} j_1 + \dots + \alpha_i^{(L)} j_L} \prod_{i=1}^{B^{(1)}} (b_i^{(1)})_{\beta_i^{(1)} j_1} \dots \prod_{i=1}^{B^{(L)}} (b_i^{(L)})_{\beta_i^{(L)} j_L} z_j^{j_1} \dots z_L^{j_L}}{\prod_{i=1}^C (c_i)_{\gamma_i^{(1)} j_1 + \dots + \gamma_i^{(L)} j_L} \prod_{i=1}^{D^{(1)}} (d_i^{(1)})_{\delta_i^{(1)} j_1} \dots \prod_{i=1}^{D^{(L)}} (d_i^{(L)})_{\delta_i^{(L)} j_L} j_1! \dots j_L!}, \quad (\text{A.8})$$

where the following notation is used:

$$[a : \alpha^{(1)}, \dots, \alpha^{(L)}] \equiv (a_1 : \alpha_1^{(1)}, \dots, \alpha_1^{(L)}), \dots, (a_A : \alpha_A^{(1)}, \dots, \alpha_A^{(L)}); \\ [b^{(M)} : \beta^{(M)}] \equiv (b_1^{(M)} : \beta_1^{(M)}), \dots, (b_{B^{(M)}}^{(M)} : \beta_{B^{(M)}}^{(M)}); \quad M = 1, \dots, L;$$

and an analogous notation for  $[c : \gamma]$  and  $[d : \delta]$ . In the formula (A.8) it is implied that all  $\alpha$ 's,  $\beta$ 's,  $\gamma$ 's and  $\delta$ 's are non-negative integer numbers.

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