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General results for massive N-point Feynman diagrams with different masses

A. I. Davydychev

Institute for Nuclear Physics of Moscow State University,
119899, Moscow, USSR

Abstract

General expressions are obtained for scalar one-loop massive Feynman diagrams with any number of external lines and with arbitrary momenta of these lines, with arbitrary values of masses and powers of denominators of internal lines, the space-time dimension n also being arbitrary. Special cases are considered when two (or more) masses are equal to zero. The results are represented in terms of hypergeometric functions.

1 Introduction

The present paper is devoted to obtaining expressions for one-loop massive Feynman diagrams of the general form. When obtaining these results, we shall use the method of evaluating massive Feynman integrals developed in Refs. [1, 2]. In fact, the present paper continues and generalizes the results of Ref. [2] since the case of arbitrary masses of "internal" particles is considered here. Therefore, we shall not repeat arguments in favour of obvious importance of such examinations, and we shall not make a review of a great number of papers devoted to evaluation of various massless and massive Feynman integrals, referring the reader to the Ref. [2]. We want to add to that reference list the papers [3, 4] where some examples of dimensionally regularized two-, three-, and four-point Feynman integrals have been considered, and the papers [5, 6] where some examples of two-loop calculations of massive diagrams have been presented.

The method of evaluating massive Feynman integrals used in the present paper is based on the representation of massive denominators in the form of the Mellin–Barnes contour integrals (see also the Appendix):

$$\frac{1}{(k^2-m^2+\mathrm{i}0)^{\nu}} = \frac{1}{\Gamma(\nu)}\,\frac{1}{2\pi\mathrm{i}}\int\limits_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}s\; \frac{(-m^2)^s}{(k^2+\mathrm{i}0)^{\nu+s}}\,\Gamma(-s)\,\Gamma(\nu+s)\;, \tag{1.1}$$

where ν is the index of the line (the power of denominator), k is the momentum, m is the mass, and imaginary infinitesimals in denominators define the usual "causal" way of dealing with singularities in the pseudo-Euclidean space. We shall below imply that all squared momenta in denominators have such additions, without writing explicitly these infinitesimals. It should be noted that the formula (1.1) corresponds to the Mellin–Barnes representation for the function ${}_1F_0(\nu|z) \equiv (1-z)^{-\nu}$ (see the Appendix). The integration contour in the complex s plane must separate "right" series of poles in the integrand ($\Gamma(-s)$) from "left" series of poles ($\Gamma(\nu+s)$). Below we shall understand all contour integrals in this sense. Using arbitrary values of the space-time dimension and of the powers of denominators (dimensional and analytic regularization schemes) enables us to avoid the appearance of multiple poles in the process of evaluation; therefore, such a contour can always be chosen. The representation (1.1) is very convenient since it can be used in the case when $|k^2| > m^2$ as well as when $|k^2| < m^2$ (see Refs. [1, 2]).

It can be noted that the Mellin transform and the Mellin–Barnes representation have been used earlier in Refs. [7, 8, 9, 10] to study some classes of integrals and diagrams (in particular, to calculate α-parametrized integrals).

The present paper is organized as follows. In Sec. 2 we illustrate the general technique through an example of evaluating two-point diagrams with different masses. In Sec. 3 general results for N-point diagrams with different masses are obtained. Section 4 is devoted to the examination of special cases when two (or more) masses are equal to zero. In Sec. 5 we show the possibility of symmetric representation of the results obtained. In Sec. 6 we formulate the main results of the paper and discuss the possibilities of applying these results. In the Appendix the necessary information about the hypergeometric functions occurring in the paper is given.

2 Two-point diagrams

As a simple example of applying the technique (1.1), let us consider the scalar one-loop two-point diagram with different masses m_1 and m_2 (see Fig. 1). The corresponding Feynman integral is of the following form:

$$J^{(2)}(\nu_1, \nu_2 |\ p;\ m_1, m_2) \equiv \int \frac{\mathrm{d}^n k}{(k^2 - m_1^2)^{\nu_1} [(p-k)^2 - m_2^2]^{\nu_2}} \ , \tag{2.1}$$

where $n = 4 - 2\varepsilon$ is the space-time dimension in the framework of dimensional regularization [11, 12] (see also the Ref. [13]). It should be noted that a particular case of the integral (2.1) (with $\nu_1 = \nu_2 = 1$) has been considered in Ref. [3].

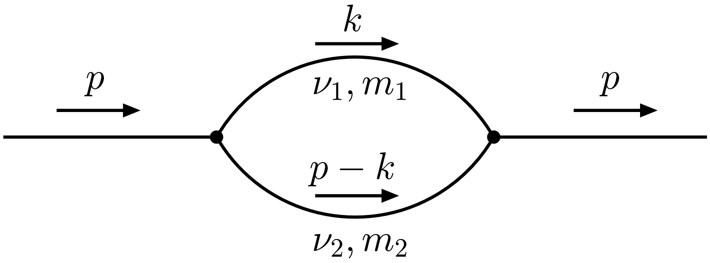

Fig. 1: The two-point diagram

Applying the formula (1.1) to both denominators of (2.1) yields

$$\begin{aligned} J^{(2)}(\nu_1, \nu_2 |\ p;\ m_1, m_2) \ &= \ \frac{1}{\Gamma(\nu_1)\ \Gamma(\nu_2)}\ \frac{1}{(2\pi \mathrm{i})^2} \int\limits_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\!\!\int \mathrm{d}s\ \mathrm{d}t\ (-m_1^2)^s (-m_2^2)^t \Gamma(-s)\ \Gamma(-t) \\ &\times \Gamma(\nu_1 + s)\ \Gamma(\nu_2 + t)\ J^{(2)}(\nu_1 + s, \nu_2 + t |\ p; 0, 0)\ , \end{aligned} \tag{2.2}$$

where the expression for the corresponding massless integral is well known:

$$J^{(2)}(\nu_1, \nu_2 |\ p; 0, 0) = \pi^{n/2}\ \mathrm{i}^{1-n}\ (p^2)^{n/2 - \nu_1 - \nu_2}\ \frac{\Gamma(n/2 - \nu_1)\ \Gamma(n/2 - \nu_2)\ \Gamma(\nu_1 + \nu_2 - n/2)}{\Gamma(\nu_1)\ \Gamma(\nu_2)\ \Gamma(n - \nu_1 - \nu_2)} \ . \tag{2.3}$$

Inserting (2.3) into (2.2) and setting $t = n/2 - \nu_1 - \nu_2 - s - u$, we find

$$\begin{aligned} J(\nu_1, \nu_2 |\ p; m_1, m_2) \ &= \ \pi^{n/2}\ \mathrm{i}^{1-n}\ (-m_2^2)^{n/2 - \nu_1 - \nu_2}\ \left[\Gamma(\nu_1)\ \Gamma(\nu_2)\right]^{-1} \\ &\times \frac{1}{(2\pi i)^2} \int\limits_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\!\!\int \mathrm{d}u\ \mathrm{d}s \left(-\frac{p^2}{m_2^2}\right)^u \left(\frac{m_1^2}{m_2^2}\right)^s \Gamma(-u)\ \Gamma(-s)\ \Gamma(n/2 - \nu_1 - s) \\ &\times \frac{\Gamma(\nu_1 + \nu_2 - n/2 + u + s)\ \Gamma(\nu_1 + u + s)}{\Gamma(n/2 + u)} \ . \end{aligned} \tag{2.4}$$

Considering the integral over s, we see that in the right half-plane of the complex variable s we have two series of poles owing to the gamma functions $\Gamma(-s)$ and

$\Gamma(n/2 \; - \; \nu_1 \; - \; s)$. By using the formula (A.7) (see also Ref. [2]) this s integral can be transformed into the integral with one series of poles in the right half-plane:

$$\begin{aligned} J(\nu_1,\nu_2|\, p; m_1,m_2) \;&=\; \pi^{n/2}\, \mathrm{i}^{1-n}\, (-m_2^2)^{n/2-\nu_1-\nu_2}\, \left[\Gamma(\nu_1)\,\Gamma(\nu_2)\right]^{-1} \\ &\times \frac{1}{(2\pi i)^2} \int\!\!\!\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}u\; \mathrm{d}s\, \left(-\frac{p^2}{m_2^2}\right)^u \left(\frac{m_1^2}{m_2^2}-1\right)^s \Gamma(-u)\,\Gamma(-s) \\ &\times \frac{\Gamma(\nu_1+\nu_2-n/2+u+s)\,\Gamma(\nu_1+u+s)\,\Gamma(\nu_2+u)}{\Gamma(\nu_1+\nu_2+2u+s)}\,. \qquad (2.5) \end{aligned}$$

In fact, we passed here from the variable m_1^2/m_2^2 to the variable $(1-m_1^2/m_2^2)$ (in Sec. 3 such a trick will enable us to obtain explicit expressions for N-point integrals).

Closing the u and s contours in (2.5) to the right we obtain the result in the form of a double hypergeometric series:

$$\begin{aligned} J(\nu_1,\nu_2|\, p; m_1,m_2) \;&=\; \pi^{n/2}\, \mathrm{i}^{1-n}\, (-m_2^2)^{n/2-\nu_1-\nu_2}\, \frac{\Gamma(\nu_1+\nu_2-n/2)}{\Gamma(\nu_1+\nu_2)} \\ &\times \sum_{j=0}^{\infty}\sum_{l=0}^{\infty} \frac{1}{j!\; l!} \left(\frac{p^2}{m_2^2}\right)^j \left(1-\frac{m_1^2}{m_2^2}\right)^l \frac{(\nu_1+\nu_2-n/2)_{j+l}(\nu_1)_{j+l}(\nu_2)_j}{(\nu_1+\nu_2)_{2j+l}}, \qquad (2.6) \end{aligned}$$

where $(\nu)_j \equiv \Gamma(\nu+j)/\Gamma(\nu)$ is the Pochhammer symbol. Here and henceforth we imply that the values of variables correspond to convergence regions of occurring hypergeometric series. The expressions in other regions can be obtained by using appropriate analytic continuation formulae for hypergeometric functions (see also the Appendix).

It should be noted that the RHS of formula (2.6) can be represented in the form of the generalized Kampé de Fériet hypergeometric function (see the Appendix):

$$\begin{aligned} J(\nu_1,\nu_2|\, p; m_1,m_2) \;&=\; \pi^{n/2}\, \mathrm{i}^{1-n}\, (-m_2^2)^{n/2-\nu_1-\nu_2}\, \frac{\Gamma(\nu_1+\nu_2-n/2)}{\Gamma(\nu_1+\nu_2)} \\ &\times F^{2:1;0}_{1:0;0}\left[\begin{array}{c} (\nu_1+\nu_2-n/2:1,1), (\nu_1:1,1) : (\nu_2:1) \\ (\nu_1+\nu_2:2,1) \end{array} \middle|\, \frac{p^2}{m_2^2}, 1-\frac{m_1^2}{m_2^2} \right]. \qquad (2.7) \end{aligned}$$

Considering the coefficients of the expansion of the function (2.7) in p^2, we can see that they can be expressed in terms of the Gauss' hypergeometric function ${}_2F_1$:

$$\begin{aligned} J(\nu_1,\nu_2|\, p; m_1,m_2) \;&=\; \pi^{n/2}\, \mathrm{i}^{1-n}\, (-m_2^2)^{n/2-\nu_1-\nu_2}\, \frac{\Gamma(\nu_1+\nu_2-n/2)}{\Gamma(\nu_1+\nu_2)} \\ &\times \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{p^2}{m_2^2}\right)^j \frac{(\nu_1+\nu_2-n/2)_j(\nu_1)_j(\nu_2)_j}{(\nu_1+\nu_2)_{2j}} \\ &\times {}_2F_1\left(\begin{array}{c} \nu_1+\nu_2-n/2+j,\; \nu_1+j \\ \nu_1+\nu_2+2j \end{array} \middle|\, 1-\frac{m_1^2}{m_2^2} \right). \qquad (2.8) \end{aligned}$$

The formulae of analytic continuation of the function ${}_2F_1$ to other variables are well known (see, e.g., Ref. [14]). Using, for example, the formula (A.4), we obtain from (2.8) the same result with the substitution $(m_1,\nu_1) \leftrightarrow (m_2,\nu_2)$. This fact confirms the "hidden" symmetry of obtained expressions (2.6)–(2.8) (see also Sec. 5, where a possibility of the explicitly symmetric representation of the results will be demonstrated).

It should be noted that from the formula (2.4) one can obtain results in the form of hypergeometric functions of the variables $(p^2/m_2^2, m_1^2/m_2^2)$ and $(m_1^2/p^2, m_2^2/p^2)$ (see Ref. [1]). These expressions correspond to analytic continuations of the function (2.6) and can be represented through the sums of the Appell's functions F_4 (see, e.g., Refs. [14, 15]). The convenience of the result of (2.6) and (2.7) is that it is represented through one hypergeometric series.

Let us consider some particular cases of the formula (2.7).

(a) If $m_1 = m_2 \equiv m$, then $1 - m_1^2/m_2^2 = 0$, and we obtain the function ${}_3F_2$ (the result coincides with the expression obtained in Ref. [1]).

(b) If $m_1 = 0$ and $m_2 \equiv m$, then, using the formula (A.3), we obtain the Gauss' function ${}_2F_1$ (the result also coincides with the Ref. [1], see also Ref. [16]).

3 N-point "sun-type" diagrams

In this section we shall show that the technique developed in Sec. 2 can without principal difficulties be generalized for diagrams with an arbitrary number of external lines. Let us consider the scalar one-loop N-point diagram given by Fig. 2. This figure is like the sun with N rays, therefore we shall call such diagrams "sun-type" diagrams.

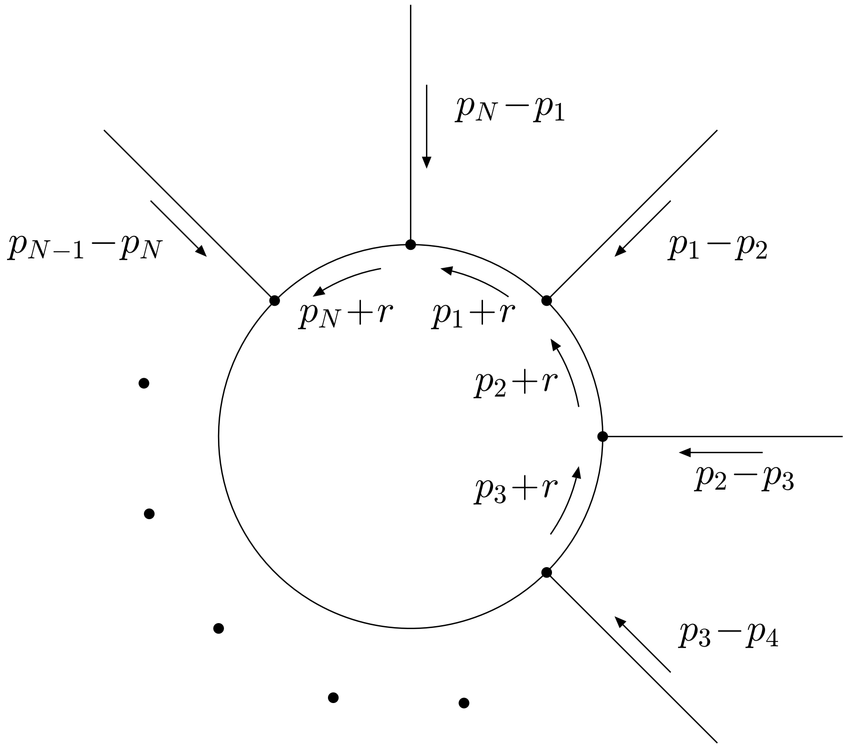

Fig. 2. The N-point "sun-type" diagram

The corresponding Feynman integral is of the following form:

$$\begin{aligned} J^{(N)}(\{\nu_j\}|\{p_j\};\{m_j\}) &\equiv \int \frac{\mathrm{d}^n r}{[(p_1+r)^2-m_1^2]^{\nu_1}\ldots[(p_N+r)^2-m_N^2]^{\nu_N}} \\ &= \int \frac{\mathrm{d}^n r}{\prod_{j=1}^{N}[(p_j+r)^2-m^2]^{\nu_j}}\ . \end{aligned} \tag{3.1}$$

In Ref. [2] the expressions for some special cases of the integral (3.1) have been obtained: (i) the case when $m_1 = \ldots = m_N \equiv m$ and (ii) the case when all ingoing momenta are zeros, i.e., $p_1 = \ldots p_N$. In the present paper we shall examine the general case of the integral (3.1).

Applying the basic formula of the method (1.1) to all N denominators of (3.1) yields

$$\begin{aligned} J^{(N)}(\{\nu_j\}|\{p_j\};\{m_j\}) &= \frac{1}{\Pi\Gamma(\nu_i)}\,\frac{1}{(2\pi\mathrm{i})^N}\int\limits_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\!\!\ldots\!\int \left\{\prod_{i=1}^{N}\mathrm{d}t_i\ (-m_i^2)^{t_i}\ \Gamma(-t_i)\ \Gamma(\nu_i+t_i)\right\} \\ &\quad\times J^{(N)}(\{\nu_j+t_j\}|\{p_j\};0)\ . \end{aligned} \tag{3.2}$$

In Ref. [2] a representation of the corresponding massless integral has been obtained in the form of the $(L-1)$-fold Mellin–Barnes integral ($L \equiv N(N-1)/2$ is the number of independent momentum invariants):

$$\begin{aligned} J^{(N)}(\{\nu_j\}|\{p_j\};0) &= \pi^{n/2}\ \mathrm{i}^{1-n}\ (k_{1N}^2)^{n/2-\Sigma\nu_i}\frac{1}{\Gamma(n-\Sigma\nu_i)\ \Pi\Gamma(\nu_i)} \\ &\times\frac{1}{(2\pi\mathrm{i})^{L-1}}\int\limits_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\!\!\ldots\!\int \prod_{\substack{j<l\\(j,l)\neq(1,N)}} \left\{\mathrm{d}s_{jl}\ \left(\frac{k_{jl}^2}{k_{1N}^2}\right)^{s_{jl}}\ \Gamma(-s_{jl})\right\} \\ &\times\Gamma\Big(\sum\nu_i-\frac{n}{2}+\ \sum_{\substack{j<l\\(j,l)\neq(1,N)}}\!\!\sum s_{jl}\Big)\prod_{i=2}^{N-1}\Gamma\Big(\nu_i+\sum_{j<i}s_{ji}+\sum_{l>i}s_{il}\Big) \\ &\times\Gamma\Big(\frac{n}{2}-\sum\nu_i+\nu_1-\sum_{\substack{j<l\\j\neq1}}\!\sum s_{jl}\Big)\ \Gamma\Big(\frac{n}{2}-\sum\nu_i+\nu_N-\sum_{\substack{j<l\\j\neq N}}\!\sum s_{jl}\Big). \end{aligned} \tag{3.3}$$

where $k_{jl} \equiv p_j - p_l$ is the total momentum ingoing between the jth and the lth segments of the ring (see Fig. 2), and Σ and Π denote the sum and the product from 1 to N (in the case when these limits are not written explicitly). In this representation we are compelled to use one of the momentum invariants, k_{1N}^2, as a dimensionless-making parameter since we do not have another massive parameter. As a result, the symmetry of this expression is broken, and the structure of poles in the integrand is rather complicated.

Inserting the representation (3.3) into (3.2) and restoring the variable s_{1N} by the substitution

$$t_N = \frac{n}{2} - \sum\nu_i - \sum_{i=1}^{N-1} t_i - \sum_{\substack{j<l\\(j,l)\neq(1,N)}}\!\!\sum s_{jl} - s_{1N}\ ,$$

we get the following $(L+N-1)$-fold representation:

$$J^{(N)}(\{\nu_j\}|\{p_j\};\{m_j\}) = \pi^{n/2}\ \mathrm{i}^{1-n}\ (-m_N^2)^{n/2-\Sigma\nu_i}\frac{1}{\Pi\Gamma(\nu_i)}\ \frac{1}{(2\pi\mathrm{i})^{L+N-1}}$$

$$
\begin{aligned}
&\times \int\limits_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\!\!\ldots\!\int \left\{ \prod_{j<l} \mathrm{d}s_{jl} \; \left(-\frac{k_{jl}^2}{m_N^2}\right)^{s_{jl}} \Gamma(-s_{jl})\right\} \left[\Gamma\left(\frac{n}{2}+\sum\sum_{j<l} s_{jl}\right)\right]^{-1} \\
&\times \int\limits_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\!\!\ldots\!\int \left\{ \prod_{i=1}^{N-1} \mathrm{d}t_i \left(\frac{m_i^2}{m_N^2}\right)^{t_i} \Gamma(-t_i)\, \Gamma\left(\nu_i+t_i+\sum_{j<i} s_{ji}+\sum_{l>i} s_{il}\right)\right\} \\
&\times \Gamma\left(\frac{n}{2}-\sum\nu_i+\nu_N-\sum\sum_{\substack{j<l\\ l\neq N}} s_{jl}-\sum_{i=1}^{N-1} t_i\right) \\
&\times \Gamma\left(\sum\nu_i-\frac{n}{2}+\sum\sum_{j<l} s_{jl}+\sum_{i=1}^{N-1} t_i\right) . \qquad (3.4)
\end{aligned}
$$

By analogy with the s integral in (2.4), each of $(N-1)$ t_i integrals in (3.4) has two series of poles in the right half-plane of the corresponding complex variable. Transforming consecutively ($N-1$ times) all t_i integrals by using the formula (A.7), we “disentangle” the expression (3.4) and obtain

$$
\begin{aligned}
J^{(N)}(\{\nu_j\}|\{p_j\};\{m_j\}) \;=\; & \pi^{n/2}\; \mathrm{i}^{1-n}\; (-m_N^2)^{n/2-\Sigma\nu_i} \frac{1}{\Pi\Gamma(\nu_i)} \\
& \times \frac{1}{(2\pi \mathrm{i})^{L+N-1}} \int\limits_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\!\!\ldots\!\int \left\{ \prod_{j<l} \mathrm{d}s_{jl} \; \left(-\frac{k_{jl}^2}{m_N^2}\right)^{s_{jl}} \Gamma(-s_{jl})\right\} \\
& \times \left\{ \prod_{i=1}^{N-1} \mathrm{d}t_i \; \left(\frac{m_i^2}{m_N^2}-1\right)^{t_i} \Gamma(-t_i)\, \Gamma\left(\nu_i+t_i+\sum_{j<i} s_{ji}+\sum_{l>i} s_{il}\right)\right\} \\
& \times \Gamma\left(\nu_N+\sum_{j<N} s_{jN}\right) \frac{\Gamma\left(\sum\nu_i-\frac{n}{2}+\sum\sum_{j<l} s_{jl}+\sum_{i=1}^{N-1} t_i\right)}{\Gamma\left(\sum\nu_i+2\sum\sum_{j<l} s_{jl}+\sum_{i=1}^{N-1} t_i\right)} . \qquad (3.5)
\end{aligned}
$$

In the representation (3.5) we have only one series of poles in the right half-plane of each of the variables s_{jl} and t_i. Therefore, closing all the contours to the right, we obtain the result in the form of one $(L+N-1)$-fold hypergeometric series,

$$
\begin{aligned}
J^{(N)}(\{\nu_j\}|\{p_j\};\{m_j\}) \;=\; & \pi^{n/2}\; \mathrm{i}^{1-n}\; (-m_N^2)^{n/2-\Sigma\nu_i} \frac{\Gamma(\Sigma\nu_i-n/2)}{\Gamma(\Sigma\nu_i)} \\
& \times \sum_{\substack{\{a_{jl}\}\\ j,l=1,\ldots,N\\ j<l}}\!\!\ldots\!\sum \;\; \sum_{\substack{\{b_i\}\\ i=1,\ldots,N-1}}\!\!\ldots\!\sum \;\; \prod_{j<l} \frac{1}{a_{jl}!} \left(\frac{k_{jl}^2}{m_N^2}\right)^{a_{jl}} \prod_{i=1}^{N-1} \frac{1}{b_i!} \left(1-\frac{m_i^2}{m_N^2}\right)^{b_i} \\
& \times \frac{(\Sigma\nu_i-n/2)_{\Sigma a_{jl}+\Sigma' b_i}}{(\Sigma\nu_i)_{2\Sigma a_{jl}+\Sigma' b_i}} \left[\prod_{i=1}^{N-1} (\nu_i)_{\Sigma_{j<i} a_{ji}+\Sigma_{l>i} a_{il}+b_i}\right] (\nu_N)_{\Sigma_{j<N} a_{jN}} , \qquad (3.6)
\end{aligned}
$$

where $\Sigma a_{jl} \equiv \sum\sum_{j<l} a_{jl}$, $\Sigma' b_i \equiv \Sigma_{i=1}^{N-1} b_i$ and $L \equiv N(N-1)/2$. The series in the RHS of (3.6) can be represented in the form of the generalized Lauricella hypergeometric function

(see the Appendix),

$$J^{(N)}(\{\nu_j\}|\{p_j\};\{m_j\}) = \pi^{n/2}\; \mathrm{i}^{1-n}\; (-m_N^2)^{n/2-\Sigma\nu_i}\, \frac{\Gamma(\Sigma\nu_i - n/2)}{\Gamma(\Sigma\nu_i)}$$

$$\times F^{N+1:0;\ldots;0}_{1\;:0;\ldots;0}\left[\begin{matrix}(\Sigma\nu_i - n/2 : L+N-1 \text{ units}), \left\{\left(\nu_i : {\scriptstyle N \text{ units}, \, L\text{–}1 \text{ zeros } (i\neq N) \atop \scriptstyle N\text{–}1 \text{ units}, \, L \text{ zeros } (i=N)}\right)\right\} \\ (\Sigma\nu_i \;:\; L \;\text{ twos},\; N-1 \;\text{ units})\end{matrix}\,\middle|\, \left\{\frac{k_{jl}^2}{m_N^2}\right\}, \left\{1-\frac{m_i^2}{m_N^2}\right\}\right], \tag{3.7}$$

where the order of arrangement of units and zeros at the parameters ν_i depends on the order of the variables and follows from the expression (3.6). The set of variables of the function (3.7) consists of two subsets: (i) momentum invariants $k_{jl}^2 \equiv (p_j - p_l)^2$ made dimensionless by m_N^2 ($L = N(N-1)/2$ variables) and (ii) "mass" variables $(1 - m_i^2/m_N^2)$, $i \neq N$ ($N-1$ variables). The total number of variables is $(L+N-1) = (N-1)(N+2)/2$. It can be noted that any other mass can be chosen as a dimensionless-making parameter (see below).

Let us present another expression for the results (3.6) and (3.7). Considering the expansion coefficients at different powers of k_{jl}^2 one can see that these coefficients can be expressed in terms of the ordinary Lauricella function of $(N-1)$ variables, $F_D^{(N-1)}$ (see the Appendix). The result can be represented in the following form:

$$J^{(N)}(\{\nu_j\}|\{p_j\};\{m_j\}) = \pi^{n/2}\; \mathrm{i}^{1-n}\; (-m_N^2)^{n/2-\Sigma\nu_i}\, \frac{\Gamma(\Sigma\nu_i - n/2)}{\Gamma(\Sigma\nu_i)}$$

$$\times \sum_{\substack{\{a_{jl}\} \\ j,l=1,\ldots,N \\ j<l}} \!\!\cdots\!\! \sum \left\{ \prod_{j<l} \frac{1}{a_{jl}!} \left(\frac{k_{jl}^2}{m_N^2}\right)^{a_{jl}} \right\} \frac{(\Sigma\nu_i - n/2)_{\Sigma a_{jl}}}{(\Sigma\nu_i)_{2\Sigma a_{jl}}} \prod_{i=1}^{N} (\nu_i)_{\Sigma_{j<i} a_{ji} + \Sigma_{l>i} a_{il}}$$

$$\times F_D^{(N-1)}\left(\Sigma\nu_i - \frac{n}{2} + \Sigma a_{jl},\; \left\{\nu_i + \sum_{j<i} a_{ji} + \sum_{l>i} a_{il}\right\}\Bigg|_{i\neq N}; \Sigma\nu_i + 2\Sigma a_{jl} \,\middle|\, \left\{1-\frac{m_i^2}{m_N^2}\right\}\right). \tag{3.8}$$

For the Lauricella function F_D some simple formulae of transformation (analytic continuation) to other variables are known (see, e.g., Ref. [15] and the Appendix). In particular, by using the formula (A.10) one can easily rewrite the result (3.8) in terms of the variables $\{1 - m_N^2/m_i^2\}$. Using the formula (A.11) we pass to the variables $\{1 - m_i^2/m_1^2\}$ ($i = 2, \ldots, N$), and we obtain, from (3.8), the result where m_1^2 (instead of m_N^2) is a dimensionless-making parameter. Analogously one can pass to any other parameter m_i^2. This fact indicates on a "hidden" symmetry of the expressions (3.6)–(3.8) with respect to the indices $1, \ldots, N$ (this symmetry is obvious for the integral (3.1)). In Sec. 5 the possibility of an explicitly symmetric representation of the results will be considered.

Let us consider some special cases of the general results (3.6)–(3.8).

(a) If all ingoing momenta are zeros, all k_{jl} vanish, and we obtain from (3.8) the result of Ref. [2] expressed in terms of the function $F_D^{(N-1)}$.

(b) If $m_1 = m_2 = \ldots = m_N \equiv m$, then all the variables of the type $\{1 - m_i^2/m_N^2\}$ vanish, the function $F_D^{(N-1)}$ in (3. 8) is equal to unity, and we come to the result of Ref. [2] represented in the form of the generalized Lauricella function of L variables. In this case, the well-known gamma function duplication formula should be taken into account, from which follows that

$$(a)_{2j} = 4^j (a/2)_j ((a+1)/2)_j \ .$$

(c) If $N = 1$, the sum in (3.6) should be replaced by unity and we obtain the well-known result for massive "tadpole" diagram [11]:

$$J^{(1)}(\nu|m) = \pi^{n/2}\ \mathrm{i}^{1-n}(-m^2)^{n/2-\nu}\ \frac{\Gamma(\nu - n/2)}{\Gamma(\nu)}\ . \qquad (3.9)$$

If $N = 2$ then the formulae (3.6)–(3.8) turn into the formulae (2.6)–(2.8).

(d) The special case when some of the masses are zeros will be examined in Sec. 4.

4 The case when some masses are equal to zero

As a rule, the values of variables equal to unity do not belong to the convergence regions of corresponding hypergeometric series. Because of this fact, we must very carefully consider the limits when some masses vanish, otherwise we can obtain a wrong result.

Let us illustrate this through an example of the three-point "triangle" diagram (see Fig. 3). Let us consider the general expression for $J^{(3)}(\{\nu_j\}|\{p_j\};\{m_j\})$ that follows from (3.6). If we put $m_1 = 0$ and $m_2 = m_3 \equiv m$, then only the term with $b_2 = 0$ survives in the sum over b_2, and summing over b_1 can be performed by using the formula (A.3). Finally, we obtain the correct result coinciding with that of Ref. [1].

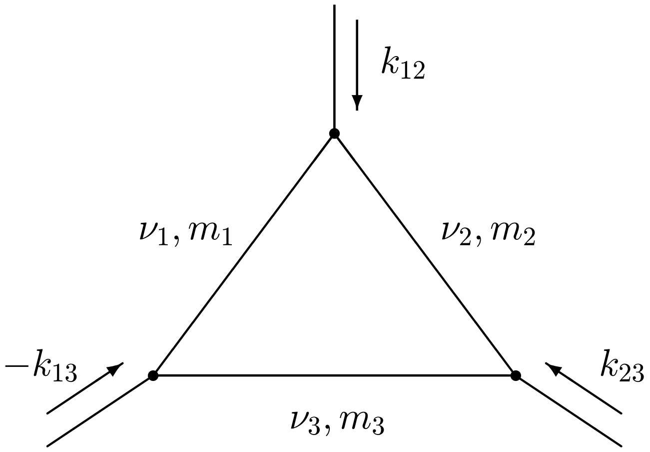

Fig. 3. Three-point "triangle" diagram

Let us examine now the case when, in the general expression (3.6) (at $N = 3$), we simultaneously formally put $m_1 = m_2 = 0$ ($m_3 \equiv m$). Then we can sum over b_1 and b_2 by using the formula (A.3), and we obtain the following result for $J^{(3)}(\{\nu_j\}|\{p_j\};0,0,m)$:

$$\pi^{n/2}\ \mathrm{i}^{1-n}\ (-m^2)^{n/2-\nu_1-\nu_2-\nu_3}\ \frac{\Gamma(\nu_1+\nu_2+\nu_3-n/2)\ \Gamma(n/2-\nu_1-\nu_2)}{\Gamma(n/2)\ \Gamma(\nu_3)}$$
$$\times \sum_{a_{12}=0}^{\infty} \sum_{a_{13}=0}^{\infty} \sum_{a_{23}=0}^{\infty} \frac{1}{a_{12}!\ a_{13}!\ a_{23}!} \left(-\frac{k_{12}^2}{m^2}\right)^{a_{12}} \left(\frac{k_{13}^2}{m^2}\right)^{a_{13}} \left(\frac{k_{23}^2}{m^2}\right)^{a_{23}}$$

$$\times \frac{(\nu_1+\nu_2+\nu_3-n/2)_{a_{12}+a_{13}+a_{23}}\;(\nu_1)_{a_{12}+a_{13}}\;(\nu_2)_{a_{12}+a_{23}}}{(n/2)_{a_{12}+a_{13}+a_{23}}\;(\nu_1+\nu_2-n/2+1)_{a_{12}}}\;. \tag{4.1}$$

One can easily see that the result (4.1) is incorrect. To do this, let us remove the third (lower) line of the triangle (Fig. 3) and join two lower vertices (i.e., the case $\nu_3 = 0$ is considered). In this case the formula (4.1) gives us zero (since $\Gamma(\nu_3)$ is in the denominator), while we must obtain here the result (2.2) with $p = k_{12}$.

To understand this discrepancy, let us consider the Mellin–Barnes representation for $J^{(3)}$ that follows from (3.5):

$$\begin{aligned}
J^{(3)}(\{\nu_j\}|\{p_j\};\{m_j\}) \;=\; & \pi^{n/2}\,\mathrm{i}^{1-n}\,(-m_3^2)^{n/2-\nu_1-\nu_2-\nu_3}\frac{1}{\Gamma(\nu_1)\,\Gamma(\nu_2)\,\Gamma(\nu_3)} \\
& \times\frac{1}{(2\pi\mathrm{i})^5}\int\limits_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\!\!\ldots\!\int \mathrm{d}s_{12}\,\mathrm{d}s_{13}\,\mathrm{d}s_{23}\,\mathrm{d}t_1\,\mathrm{d}t_2 \\
& \times\left(-\frac{k_{12}^2}{m_3^2}\right)^{s_{12}}\left(-\frac{k_{13}^2}{m_3^2}\right)^{s_{13}}\left(-\frac{k_{23}^2}{m_3^2}\right)^{s_{23}}\left(\frac{m_1^2}{m_3^2}-1\right)^{t_1}\left(\frac{m_2^2}{m_3^2}-1\right)^{t_2} \\
& \times\Gamma(-s_{12})\,\Gamma(-s_{13})\,\Gamma(-s_{23})\,\Gamma(-t_1)\,\Gamma(-t_2) \\
& \times\Gamma(\nu_1+s_{12}+s_{13}+t_1)\,\Gamma(\nu_2+s_{12}+s_{23}+t_2)\,\Gamma(\nu_3+s_{13}+s_{23}) \\
& \times\frac{\Gamma(\nu_1+\nu_2+\nu_3-n/2+s_{12}+s_{13}+s_{23}+t_1+t_2)}{\Gamma(\nu_1+\nu_2+\nu_3+2s_{12}+2s_{13}+2s_{23}+t_1+t_2)}.
\end{aligned} \tag{4.2}$$

Putting $m_1 = m_2 = 0$ ($m_3 \equiv m$) in (4.2) and applying twice the formula (A.6) for integration with respect to t_1 and t_2, we find

$$\begin{aligned}
J^{(3)}(\{\nu_j\}|\{p_j\};0,0,m) = {} & \pi^{n/2}\,\mathrm{i}^{1-n}\,(-m^2)^{n/2-\nu_1-\nu_2-\nu_3}\frac{1}{\Gamma(\nu_1)\,\Gamma(\nu_2)\,\Gamma(\nu_3)} \\
& \times\frac{1}{(2\pi\mathrm{i})^3}\iiint\limits_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\mathrm{d}s_{12}\,\mathrm{d}s_{13}\,\mathrm{d}s_{23}\left(-\frac{k_{12}^2}{m^2}\right)^{s_{12}}\left(-\frac{k_{13}^2}{m^2}\right)^{s_{13}}\left(-\frac{k_{23}^2}{m^2}\right)^{s_{23}} \\
& \times\Gamma(-s_{12})\,\Gamma(-s_{13})\,\Gamma(-s_{23})\,\Gamma(n/2-\nu_1-\nu_2-s_{12}) \\
& \times\frac{\Gamma(\nu_1+\nu_2+\nu_3-n/2+s_{12}+s_{13}+s_{23})\,\Gamma(\nu_1+s_{12}+s_{13})\,\Gamma(\nu_2+s_{12}+s_{23})}{\Gamma(n/2+s_{12}+s_{13}+s_{23})}.
\end{aligned} \tag{4.3}$$

We see that in (4.3) an additional series of poles ($\Gamma(n/2-\nu_1-\nu_2-s_{12})$) in the right half-plane of the variable s_{12} has appeared, which should be taken into account when evaluating the corresponding integral (if only one mass vanishes such additional poles do not appear). As a result we obtain, in addition to (4.1), the second term equal to

$$\begin{aligned}
& \pi^{n/2}\,\mathrm{i}^{1-n}\,(-m^2)^{-\nu_3}(k_{12}^2)^{n/2-\nu_1-\nu_2}\,\frac{\Gamma(n/2-\nu_1)\,\Gamma(n/2-\nu_2)\,\Gamma(\nu_1+\nu_2-n/2)}{\Gamma(\nu_1)\,\Gamma(\nu_2)\,\Gamma(n-\nu_1-\nu_2)} \\
& \times\sum_{a_{12}=0}^{\infty}\sum_{a_{13}=0}^{\infty}\sum_{a_{23}=0}^{\infty}\frac{1}{a_{12}!\,a_{13}!\,a_{23}!}\left(-\frac{k_{12}^2}{m^2}\right)^{a_{12}}\left(\frac{k_{13}^2}{m^2}\right)^{a_{13}}\left(\frac{k_{23}^2}{m^2}\right)^{a_{23}} \\
& \times\frac{(\nu_3)_{a_{12}+a_{13}+a_{23}}\;(n/2-\nu_2)_{a_{12}+a_{13}}\;(n/2-\nu_1)_{a_{12}+a_{23}}}{(n-\nu_1-\nu_2)_{a_{12}+a_{13}+a_{23}}\;(n/2-\nu_1-\nu_2+1)_{a_{12}}}\;.
\end{aligned} \tag{4.4}$$

Finally, the correct result is

$$J^{(3)}(\{\nu_j\}|\{p_j\};0,0,m) = (4.1) + (4.4)\ . \tag{4.5}$$

This result coincides with the expression obtained in Ref. [1]. It should be noted that at $\nu_3 = 0$ we obtain from (4.4) a correct transition to the formula (2.2). It can be also noted that, if from the beginning (before the evaluation of the integral) we formally put $k_{12}^2 = 0$, then in the corresponding Mellin–Barnes representation the integration with respect to s_{12} will be absent (see, e.g., Ref. [2]) and the terms of the type of (4.4) will not appear. However one must also very carefully deal with putting various conditions before the evaluation of Feynman integrals (see, e.g., Ref. [17]).

The situation for arbitrary N is quite analogous to the considered situation at $N = 3$. If only one mass vanishes, additional poles do not appear in the representation (3.5) and we can use the results (3.6)–(3.8). If some two masses, m_{i_1} and m_{i_2} ($i_1, i_2 \neq N$), are equal to zero then, using the formula (A.6) to integrate with respect to t_{i_1} and t_{i_2}, we see that additional poles appear in the right half-plane of the variable $s_{i_1 i_2}$, owing to the gamma function

$$\Gamma\Big(\frac{n}{2} - \nu_{i_1} - \nu_{i_2} + \sum\sum_{j<l} s_{jl} - \sum_{j<i_1} s_{ji_1} - \sum_{l>i_1} s_{i_1 l} - \sum_{j<i_2} s_{ji_2} - \sum_{l>i_2} s_{i_2 l}\Big)\ .$$

Analogously we obtain that, if $m_{i_1} = \ldots = m_{i_k} = 0$ ($i_1, \ldots, i_k \neq N$), additional poles in the variables $s_{i_k i_{k'}}$ ($k, k' = 1, ..., K$) appear owing to the gamma function

$$\Gamma\Big(\frac{n}{2} - \sum_{k=1}^{K} \nu_{i_k} + \sum\sum_{j<l} s_{jl} - \sum_{k=1}^{K}\Big(\sum_{j<i_k} s_{ji_k} + \sum_{l>i_k} s_{i_k l}\Big)\Big)\ .$$

However, because of the fact that in the case $N > 3$ other s variables are "tangled", it is rather complicated to represent the result in the form of known hypergeometric functions.

The given examination leads to a rather interesting consequence. When considering vertex diagrams with light "internal" particles (for example, with light quarks), one cannot always formally put the masses of these particles equal to zero since the value $m = 0$ is a singular case. One must first evaluate the corresponding diagrams at $m \neq 0$, and only then one should consider the limit $m \to 0$.

5 Symmetric representation of the results

In this section we shall return to general formulae for the diagrams with different masses, and we shall show that these results can be represented in a more symmetric form if an "external" dimensional value Λ is used. This can be convenient if it is known that the theory contains a dimensional constant giving some scale (for example, Λ_{QCD}). On the other hand, it should be noted that the parameter Λ can be constructed in a symmetric way from "internal" masses: for example, $\Lambda^2 = m_1^2 + \ldots + m_N^2$.

Let us note that the main formula of the method (1.1) can be written in a more general form:

$$\frac{1}{(k^2 - m^2 + \mathrm{i}0)^\nu} = \frac{1}{\Gamma(\nu)}\,\frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}s\, \frac{(\Lambda^2 - m^2)^s}{(k^2 - \Lambda^2 + \mathrm{i}0)^{\nu+s}}\, \Gamma(-s)\, \Gamma(\nu + s)\ , \tag{5.1}$$

where Λ is an arbitrary massive parameter. One can easily see that the RHS of (5.1) does not depend on Λ and is equal to the LHS. In particular, if $\Lambda = 0$ the formula (5.1) coincides with the formula (1.1).

By using the formula (5.1), let us consider again the evaluation of the N-point integral (3.1) with different masses (see Fig. 2). Applying (5.1) to N denominators of (3.1) yields

$$\begin{aligned} J^{(N)}(\{\nu_j\}|\{p_j\};\{m_j\}) &= \frac{1}{\Pi\Gamma(\nu_i)}\,\frac{1}{(2\pi\mathrm{i})^N}\int\limits_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\!\!\ldots\!\int\left\{\prod_{i=1}^{N}\mathrm{d}t_i\;(\Lambda^2-m_i^2)^{t_i}\;\Gamma(-t_i)\;\Gamma(\nu_i+t_i)\right\} \\ &\quad\times J^{(N)}(\{\nu_j+t_j\}|\{p_j\};\Lambda)\;. \end{aligned} \tag{5.2}$$

where the following L-fold Mellin–Barnes representation [2] can be written for the integral with equal masses (see also (3.5) and (3.6)):

$$\begin{aligned} J^{(N)}(\{\nu_j\}|\{p_j\};\Lambda) &= \pi^{n/2}\;\mathrm{i}^{1-n}\;(-\Lambda^2)^{n/2-\Sigma\nu_i}\;\frac{1}{\Pi\Gamma(\nu_i)} \\ &\quad\times\frac{1}{(2\pi\mathrm{i})^L}\int\limits_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\!\!\ldots\!\int\left\{\prod_{j<l}\mathrm{d}s_{jl}\;\left(-\frac{k_{jl}^2}{\Lambda^2}\right)^{s_{jl}}\;\Gamma(-s_{jl})\right\} \\ &\quad\times\Gamma\Big(\sum\nu_i-\frac{n}{2}+\sum_{j<l}\!\sum s_{jl}\Big)\left[\Gamma\Big(\sum\nu_i+2\sum_{j<l}\!\sum s_{jl}\Big)\right]^{-1} \\ &\quad\times\prod_{i=1}^{N}\Gamma\Big(\nu_i+\sum_{j<i}s_{ji}+\sum_{l>i}s_{il}\Big)\;, \end{aligned} \tag{5.3}$$

where $L \equiv N(N-1)/2$. Inserting (5.3) into (5.2) we obtain the $(L+N)$-fold symmetric Mellin–Barnes representation for the integral (3.1),

$$\begin{aligned} J^{(N)}(\{\nu_j\}|\{p_j\};\{m_j\}) &= \pi^{n/2}\;\mathrm{i}^{1-n}\;(-\Lambda^2)^{n/2-\Sigma\nu_i}\,\frac{1}{\Pi\Gamma(\nu_i)} \\ &\times\frac{1}{(2\pi\mathrm{i})^{L+N}}\int\limits_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\!\!\ldots\!\int\left\{\prod_{j<l}\mathrm{d}s_{jl}\;\left(-\frac{k_{jl}^2}{\Lambda^2}\right)^{s_{jl}}\;\Gamma(-s_{jl})\right\}\left\{\prod_{i=1}^{N}\mathrm{d}t_i\;\left(\frac{m_i^2}{\Lambda^2}-1\right)^{t_i}\;\Gamma(-t_i)\right\} \\ &\times\frac{\Gamma\Big(\sum\nu_i-n/2+\sum\limits_{j<l}\!\sum s_{jl}+\sum t_i\Big)}{\Gamma\Big(\sum\nu_i+2\sum\limits_{j<l}\!\sum s_{jl}+\sum t_i\Big)}\;\prod\Gamma\Big(\nu_i+t_i+\sum_{j<i}s_{ji}+\sum_{l>i}s_{il}\Big)\;. \end{aligned} \tag{5.4}$$

From here one can obtain the result in the form of the $(L+N)$-fold hypergeometric series,

$$\begin{aligned} J^{(N)}(\{\nu_j\}|\{p_j\};\{m_j\}) &= \pi^{n/2}\;\mathrm{i}^{1-n}\;(-\Lambda^2)^{n/2-\Sigma\nu_i}\,\frac{\Gamma(\Sigma\nu_i-n/2)}{\Gamma(\Sigma\nu_i)} \\ &\quad\times\sum_{\substack{\{a_{jl}\}\\ j,l=1,\ldots,N\\ j<l}}\!\!\cdots\sum\;\sum_{\substack{\{b_i\}\\ i=1,\ldots,N}}\!\!\cdots\sum\;\prod_{j<l}\frac{1}{a_{jl}!}\left(\frac{k_{jl}^2}{\Lambda^2}\right)^{a_{jl}}\;\prod_{i=1}^{N}\frac{1}{b_i!}\left(1-\frac{m_i^2}{\Lambda^2}\right)^{b_i} \\ &\quad\times\frac{(\Sigma\nu_i-n/2)_{\Sigma a_{jl}+\Sigma b_i}}{(\Sigma\nu_i)_{2\Sigma a_{jl}+\Sigma b_i}}\;\prod_{i=1}^{N}(\nu_i)_{\Sigma_{j<i}a_{ji}+\Sigma_{l>i}a_{il}+b_i}\;, \end{aligned} \tag{5.5}$$

or in the form of the generalized Lauricella hypergeometric function of $(L+N)$ variables (see the Appendix):

$$J^{(N)}(\{\nu_j\}|\{p_j\};\{m_j\}) = \pi^{n/2}\;\mathrm{i}^{1-n}\;(-\Lambda^2)^{n/2-\Sigma\nu_i}\;\frac{\Gamma(\Sigma\nu_i-n/2)}{\Gamma(\Sigma\nu_i)}$$
$$\times F^{N+1:0;\ldots;0}_{\;1\;:0;\ldots;0}\left[\begin{matrix}(\Sigma\nu_i - n/2 : L{+}N \text{ units}), \{(\nu_i : N \text{ units},\ L \text{ zeros})\}\\ (\Sigma\nu_i\ :\ L\ \text{ twos},\ N\ \text{ units})\end{matrix}\left|\left\{\frac{k_{jl}^2}{\Lambda^2}\right\}, \left\{1 - \frac{m_i^2}{\Lambda^2}\right\}\right.\right], \quad (5.6)$$

The coefficients at various powers of k_{jl}^2 can be expressed in terms of the ordinary Lauricella function of N variables, $F_D^{(N)}$ (see the Appendix):

$$J^{(N)}(\{\nu_j\}|\{p_j\};\{m_j\}) = \pi^{n/2}\;\mathrm{i}^{1-n}\;(-\Lambda^2)^{n/2-\Sigma\nu_i}\frac{\Gamma(\Sigma\nu_i-n/2)}{\Gamma(\Sigma\nu_i)}$$
$$\times \sum_{\substack{\{a_{jl}\}\\ j,l=1,\ldots,N\\ j<l}}\!\!\!\ldots\sum \left\{\prod_{j<l}\frac{1}{a_{jl}!}\left(\frac{k_{jl}^2}{\Lambda^2}\right)^{a_{jl}}\right\}\frac{(\Sigma\nu_i - n/2)_{\Sigma a_{jl}}}{(\Sigma\nu_i)_{2\Sigma a_{jl}}}\prod_{i=1}^{N}(\nu_i)_{\Sigma_{j<i}a_{ji}+\Sigma_{l>i}a_{il}}$$
$$\times F_D^{(N)}\left(\Sigma\nu_i - \frac{n}{2} + \Sigma a_{jl},\ \left\{\nu_i + \sum_{j<i} a_{ji} + \sum_{l>i} a_{il}\right\};\ \Sigma\nu_i + 2\Sigma a_{jl}\left|\left\{1 - \frac{m_i^2}{\Lambda^2}\right\}\right.\right). \quad (5.7)$$

The formulae (5.4)–(5.7) give us the completely symmetric (with respect to the indices $1,\ldots,N$) representation of the results for the integral (3.1) which corresponds to the diagram of Fig. 2. A pay for the symmetry consists in necessity of introducing an "external" parameter Λ (which, however, can be constructed from $m_1,...,m_N$) and in appearance of a "superfluous" variable in hypergeometric functions (5.6) and (5.7). It should be noted that the dependence of the RHS's of (5.5)–(5.7) upon the parameter Λ is fictitious: this fact can be easily verified with the help of differentiation with respect to Λ^2.

If we choose one of the masses (e.g., m_N) as a parameter Λ, the symmetry will be broken and the formulae (5.4)–(5.7) will turn into the formulae (3.5)–(3.8).

It can be noted that the known transformation formulae for the function F_D give us a possibility to rewrite the result (5.7) in terms of other variables (for example, by using the formula (A.10) we can pass to the variables $\{1-\Lambda^2/m_i^2\}$).

6 Conclusion

Using the developed method of evaluating massive Feynman integrals [1, 2], in the present paper we have obtained the general exact expressions for the one-loop scalar N-point Feynman diagrams (see Fig. 2). The momenta of external lines, the masses of internal particles, and the powers of denominators, as well as the space-time dimension, were considered as arbitrary values. Therefore, the obtained results can be applied by using both dimensional and analytic regularization schemes. In the present paper the result of Ref. [2] (concerning the scalar diagrams with equal masses of internal lines) has been generalized. Note that any integral with the tensor structure in the numerator can be reduced to scalar integrals by using the formulae of the type of the substitutions given in Ref. [18].

The obtained results are represented in the form of generalized hypergeometric functions, the expansion coefficients of which are rather simple (see the Appendix). Experience shows that such representations are rather convenient for the algorithmization of calculations. In addition, in some cases one can use appropriate analytic continuation formulae for hypergeometric functions and study various regions of values of momenta and masses.

These results can be applied, for example, to examination of one-loop contributions to the processes $\{N_1 \text{ particles} \rightarrow N_2 \text{ particles}\}$ $(N_1 + N_2 = N)$. The obtained expressions can be used as "blocks" in appropriate multiloop calculations. In the near future we intend to apply these results to studying some processes in QCD and in electroweak model.

The presented consideration shows that the used technique of evaluating massive Feynman diagrams is rather powerful and gives us a possibility to obtain general results for classes of Feynman integrals. We hope that this method (perhaps in combination with the integration-by-parts technique [19]) will enable us to obtain some new results for multiloop massive diagrams.

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A Definitions and some properties of hypergeometric functions

In this Appendix we shall give definitions of hypergeometric functions occurring in the present paper and some properties of these functions. More detailed information about these functions can be found in Refs. [14, 15, 20, 21, 22]. We consider that the values of variables correspond to convergence regions of corresponding hypergeometric series. The expressions in some other regions of the variables can be obtained by using appropriate analytic continuation formulae.

The well-known Gauss' hypergeometric function ${}_2F_1$ is defined by

$${}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix} \,\middle|\, z \right) = \sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{(c)_j} \, \frac{z^j}{j!} \ , \tag{A.1}$$

where $(a)_j \equiv \Gamma(a+j)/\Gamma(a)$ is the Pochhammer symbol. In particular, if $b = c$ we obtain from (A.1) an ordinary powerlike function:

$${}_1F_0(a|\ z) \equiv {}_2F_1\left(\begin{matrix} a,\ b \\ b \end{matrix} \,\middle|\, z \right) = \frac{1}{(1-z)^a} \ . \tag{A.2}$$

The value of the function ${}_2F_1$ with unit argument is

$${}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix} \,\middle|\, 1 \right) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)} \ . \tag{A.3}$$

The transformation (analytic continuation) formulae for ${}_2F_1$ are well known. They enable us to pass to other variables: $1/z$, $1-z$, $1/(1-z)$, $z/(z-1)$, etc. (see Ref. [14]). For example, one of the simplest transformation formulae has the form

$$
{}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix} \,\middle|\, z \right) = (1-z)^{-a}\, {}_2F_1\left(\begin{matrix} a,\ c-b \\ c \end{matrix} \,\middle|\, \frac{z}{z-1} \right) . \tag{A.4}
$$

The Mellin–Barnes representation for the function ${}_2F_1$ can be written as

$$
{}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix} \,\middle|\, z \right) = \frac{\Gamma(c)}{\Gamma(a)\,\Gamma(b)}\, \frac{1}{2\pi \mathrm{i}} \int\limits_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}s\ (-z)^s\, \Gamma(-s)\, \frac{\Gamma(a+s)\,\Gamma(b+s)}{\Gamma(c+s)} , \tag{A.5}
$$

where the integration contour separates the "right" series of poles of gamma functions in the integrand from the "left" series. In particular, if $b = c$ we obtain the representation for the function (A.2). It also follows from the formula (A.3) that

$$
\frac{1}{2\pi \mathrm{i}} \int\limits_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}s\ (-1)^s\, \Gamma(-s)\, \frac{\Gamma(a+s)\,\Gamma(b+s)}{\Gamma(c+s)} = \frac{\Gamma(a)\,\Gamma(b)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)} . \tag{A.6}
$$

The analytic continuation formulae can also easily be represented in terms of the Mellin–Barnes integrals. Passing from the variable z to $1/z$ corresponds to closing the integration contour in (A.5) at the infinity to the left rather than to the right. It should be noted that this fact illustrates the convenience of using the representation (A.5): it is correct at $|z| < 1$ as well as at $|z| > 1$, we must only close the integration contour in the appropriate way to ensure a sufficient decreasing of the integrand at the infinity. The formula of passing from the variable z to $(1-z)$ is of the following form (see, e.g., Ref. [2]):

$$
\begin{aligned}
&\frac{1}{2\pi \mathrm{i}} \int\limits_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}s\ z^s\, \Gamma(-s)\, \Gamma(c-s)\, \Gamma(a+s)\, \Gamma(b+s) \\
&\qquad = \Gamma(a+c)\,\Gamma(b+c)\, \frac{1}{2\pi \mathrm{i}} \int\limits_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}s\ (z-1)^s\, \frac{\Gamma(-s)\,\Gamma(a+s)\,\Gamma(b+s)}{\Gamma(a+b+c+s)} .
\end{aligned} \tag{A.7}
$$

Some other useful properties of such contour integrals are given in Ref. [20].

It should be also noted that the generalized hypergeometric function of one variable is defined by

$$
{}_AF_C\left(\begin{matrix} a_1, \dots, a_A \\ c_1, \dots, c_C \end{matrix} \,\middle|\, z \right) = \sum_{j=0}^{\infty} \frac{(a_1)_j \dots (a_A)_j}{(c_1)_j \dots (c_C)_j}\, \frac{z^j}{j!} , \tag{A.8}
$$

The representation (A.5) can easily be generalized for the function ${}_AF_C$.

The Lauricella hypergeometric function of N variables $F_D^{(N)}$ is defined by

$$
F_D^{(N)}(a, b_1, \dots, b_N; c|\ z_1, \dots, z_N) = \sum_{j_1=0}^{\infty} \dots \sum_{j_N=0}^{\infty} \frac{(a)_{j_1+\dots+j_N} (b_1)_{j_1} \dots (b_N)_{j_N}}{(c)_{j_1+\dots+j_N}}\, \frac{z_1^{j_1} \dots z_N^{j_N}}{j_1!\ \dots\ j_N!} . \tag{A.9}
$$

For $N=2$ the function $F_D^{(2)}$ corresponds to the Appell's function F_1 [15], and for $N=1$ $F_D^{(1)}$ corresponds to the function ${}_2F_1$. Some simple transformation formulae are also known for the function $F_D^{(N)}$ (see Ref. [15]); for example,

$$\begin{aligned}
& F_D^{(N)}(a, b_1, \ldots, b_N; c |\ z_1, \ldots, z_N) \\
& = (1-z_1)^{-b_1} \ldots (1-z_N)^{-b_N}\ F_D^{(N)}\left(c-a, b_1, \ldots, b_N; c \;\middle|\; \frac{z_1}{z_1-1}, \ldots, \frac{z_N}{z_N-1}\right), \qquad \text{(A.10)} \\
& = (1-z_1)^{-a}\ F_D^{(N)}\left(a, c-b_1-\ldots-b_N, b_2, \ldots, b_N; c \;\middle|\; \frac{z_1}{z_1-1}, \frac{z_1-z_2}{z_1-1}, \ldots, \frac{z_1-z_N}{z_1-1}\right). \qquad \text{(A.11)}
\end{aligned}$$

The generalized Lauricella function of N variables has been introduced in Ref. [23] (see also Ref. [21]). It is of the following form:

$$\begin{aligned}
& F^{A:B^{(1)};\ldots;B^{(N)}}_{C:D^{(1)};\ldots;D^{(N)}}\left[\begin{array}{c} [a:\alpha^{(1)},\ldots,\alpha^{(N)}] : [b^{(1)}:\beta^{(1)}];\ldots;[b^{(N)}:\beta^{(N)}] \\ [c:\gamma^{(1)},\ldots,\gamma^{(N)}] : [d^{(1)}:\delta^{(1)}];\ldots;[d^{(N)}:\delta^{(N)}] \end{array} \;\middle|\; z_1,\ldots,z_N \right] \\
& = \sum_{j_1=0}^{\infty} \cdots \sum_{j_N=0}^{\infty} \frac{\prod\limits_{i=1}^{A} (a_i)_{\alpha_i^{(1)} j_1 + \ldots + \alpha_i^{(N)} j_N} \prod\limits_{i=1}^{B^{(1)}} (b_i^{(1)})_{\beta_i^{(1)} j_1} \cdots \prod\limits_{i=1}^{B^{(N)}} (b_i^{(N)})_{\beta_i^{(N)} j_N}}{\prod\limits_{i=1}^{C} (c_i)_{\gamma_i^{(1)} j_1 + \ldots + \gamma_i^{(N)} j_N} \prod\limits_{i=1}^{D^{(1)}} (d_i^{(1)})_{\delta_i^{(1)} j_1} \cdots \prod\limits_{i=1}^{D^{(N)}} (d_i^{(N)})_{\delta_i^{(N)} j_N}} \frac{z_j^{j_1} \ldots z_N^{j_N}}{j_1! \ldots j_N!}, \qquad \text{(A.12)}
\end{aligned}$$

where the following notation is used:

$$\begin{aligned}
[a:\alpha^{(1)},\ldots,\alpha^{(N)}] \;&\equiv\; (a_1:\alpha_1^{(1)},\ldots,\alpha_1^{(N)}),\ldots,(a_A:\alpha_A^{(1)},\ldots,\alpha_A^{(N)}); \\
[b^{(M)}:\beta^{(M)}] \;&\equiv\; (b_1^{(M)}:\beta_1^{(M)}),\ldots,(b_{B^{(M)}}^{(M)}:\beta_{B^{(M)}}^{(M)}); \quad M=1,\ldots,N;
\end{aligned}$$

and analogously for $[c:\gamma]$ and $[d:\delta]$. In the formula (A.12) it is implied that all α's, β's, γ's, and δ's are non-negative integers (although this formula can be generalized for the case of any non-negative numbers [21, 23]). Note that at $N=2$ the function (A.12) is often called as the generalized Kampé de Fériet function.

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