

Progress in evaluating some complicated types of Feynman diagrams with two (and more) loops

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Abstract

Problems occurring in physically important non-trivial examples of loop calculations are discussed. A procedure of deriving expansions of two-loop self-energy diagrams with different masses is constructed. The cases of small and large external momentum are considered. The coefficients of the expansions are calculated analytically. Comparison with numerical results shows good agreement below the first threshold and above the last threshold of the diagram. An approach to evaluating three- and four-point ladder diagrams with massless internal particles and off-shell external momenta is presented. Exact results for an arbitrary number of loops are obtained. Evaluating non-planar diagrams is also discussed.

In: Proc. Workshop "AIHENP-93", Oberammergau, Germany, October 1993
(World Scientific, Singapore, 1994), p. 531-537

In the present talk I would like to discuss some recent results for loop diagrams obtained in collaboration with J.B. Tausk [1, 2] V.A. Smirnov [2] and N.I. Ussyukina [3, 4, 5]. The papers [1, 2] are devoted to the examination of massive two-loop self-energy diagrams, whilst refs. [3, 4, 5] deal with some three- and four-point diagrams with massless internal particles and off-shell external momenta.

1. Two-loop massive self-energy diagrams

It is known that any two-loop self-energy diagram (see Fig. 1) can be reduced to scalar integrals [6]. Moreover, in the case when the powers of denominators are integer, the diagram in Fig. 1b can be reduced to Fig. 1a by the decomposition of the first and the fourth denominators. So, we can express all two-loop two-point contributions in terms of the scalar integrals corresponding to Fig. 1a,

$$J(\{\nu_i\}; \{m_i\}; k) = \int \int \frac{d^n p d^n q}{D_1^{\nu_1} D_2^{\nu_2} D_3^{\nu_3} D_4^{\nu_4} D_5^{\nu_5}}, \quad (1)$$

where $D_i = (p_i^2 - m_i^2 + i0)$ are massive denominators, ν_i are the powers of these denominators, $n = 4 - 2\varepsilon$ is the space-time dimension [7] and p_i are constructed from the external momentum k and the loop integration momenta p and q (with due account of the momentum conservation).

For some special cases the integrals (1) can be evaluated exactly (see, e.g., in [8, 9]) and expressed in terms of the polylogarithms Li_N (see [10]) with $N \leq 3$. On the other hand, the problem of evaluating such diagrams when all the internal lines are massive is more complicated, and exact expressions are not known. For example, in [11] the result was presented in terms of a two-fold integral representation.

In the papers [1, 2] we studied expansions of massive two-loop two-point diagrams for the case of small [1] and large [2] values of the external momentum squared. Note that in general the diagram in Fig. 1a has four physical thresholds (with respect to k^2), corresponding to different cuts of the diagram; namely,

$$(m_1 + m_4)^2, \quad (m_2 + m_5)^2, \quad (m_1 + m_3 + m_5)^2, \quad (m_2 + m_3 + m_4)^2.$$

The cases considered correspond to the situations when we are either below the lowest threshold [1] or above the highest one [2].

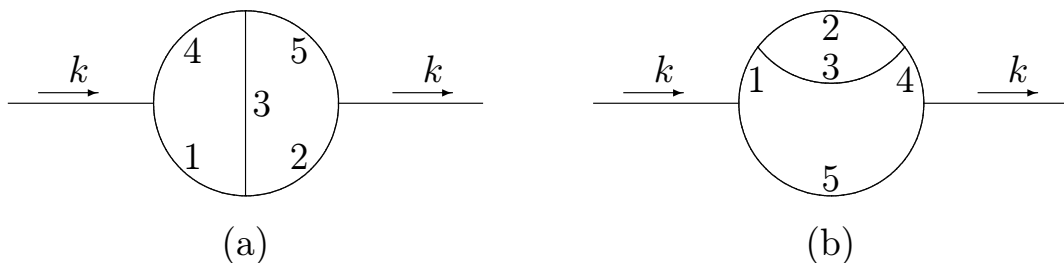


Figure 1: Two-loop self-energy diagrams

For small values of k^2 , the expansion of the integral (1) is a usual Taylor expansion (we consider the general case, when the lowest threshold is not equal to zero). Coefficients of the expansion can be represented in terms of two-loop massive “vacuum” diagrams. Moreover, for the case when all powers of denominators are integer, they can be reduced to vacuum diagrams, each of them depending no more than on three different masses (see Fig. 2). These diagrams with higher (integer) powers of denom-

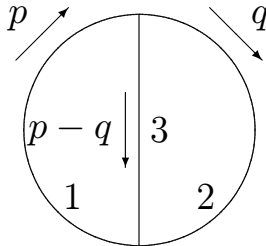


Figure 2: The vacuum two-loop diagram

inators (occurring in the coefficients of the expansion) can be evaluated by use of the integration-by-parts technique [12] (see also [13]), and all of them (up to finite in ϵ parts) can be represented in terms of dilogarithms or Clausen’s function (see [10]). By use of the REDUCE system [14], we constructed an algorithm of analytical evaluation of the coefficients of the expansion. We compared the results for some diagrams with a numerical calculation based on the integral representation [11], and we found a nice agreement in the region below the lowest threshold.

The case when k^2 is larger than the highest threshold of the diagram is more complicated, because the corresponding expansion is not a usual Taylor expansion, but also contains logarithms and squared logarithms of $-k^2$ (in four dimensions) yielding an imaginary part when the momentum is time-like. To obtain this expansion, we applied a general mathematical theorem on asymptotic expansions of Feynman integrals in the limit of large external momentum. Expansions of this kind were presented in refs. [15]. For our case (1), the asymptotic expansion theorem gives

$$J_\Gamma \underset{k^2 \rightarrow \infty}{\sim} \sum_\gamma J_{\Gamma/\gamma} \circ \mathcal{T}_{\{m_i\};\{q_i\}} J_\gamma, \quad (2)$$

where Γ is the main graph (see Fig.1a), γ are subgraphs involved in the asymptotic expansion (see below), Γ/γ is the reduced graph obtained from Γ by shrinking the subgraph γ to a single point, J_γ denotes the dimensionally-regularized Feynman integral corresponding to a graph γ (for example, J_Γ is given by (1)), $\mathcal{T}_{\{m_i\};\{q_i\}}$ is the operator of Taylor expansion of the integrand in masses and “small” momenta q_i (that are “external” for the given subgraph γ , but do not contain the “large” external momentum k), and the symbol “ \circ ” means that the resulting polynomial in these momenta should be inserted into the numerator of the integrand of $J_{\Gamma/\gamma}$. It is implied that the operator \mathcal{T} acts on the integrands before the loop integrations are performed.

In our case (see Fig. 1a), the sum (2) goes over all subgraphs γ that become one-particle irreducible when we connect the two vertices with external momentum k by

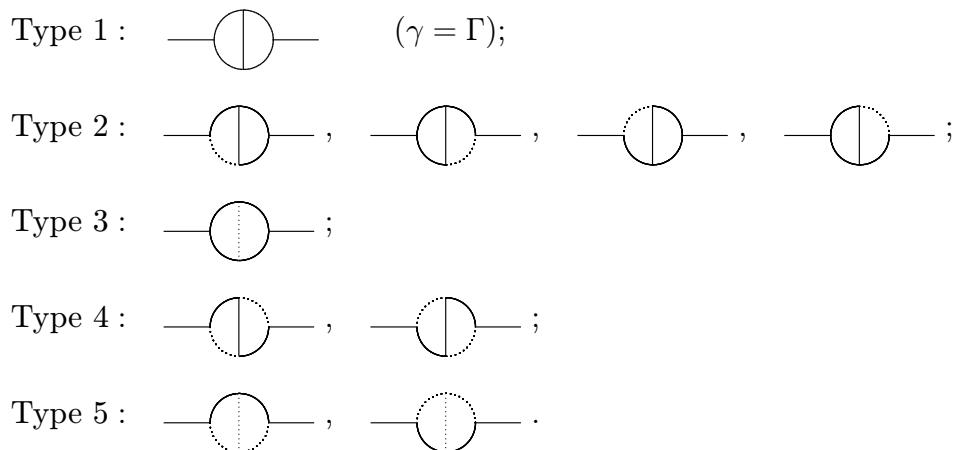


Figure 3: The subgraphs γ contributing to the large k^2 expansion

a line. These subgraphs (there are five different types of them) are shown in Fig. 3 (dotted lines correspond to the lines that do not belong to γ).

The reduced graphs Γ/γ correspond to the dotted lines and can be obtained by shrinking all solid lines to a point. In such a way, we obtain that for the second and third type (see Fig. 3) $J_{\Gamma/\gamma}$ corresponds to a massive tadpole, for the fourth type we obtain a product of two massive tadpoles, while for the fifth type we get a two-loop massive vacuum integral with three internal lines, corresponding to Fig. 2. All other integrals (occurring in separate terms of the expansion) can also be evaluated.

We have realized this algorithm also by use of the REDUCE system [14]. The analytical expressions for the coefficients contain powers and logarithms of the masses and the external momentum squared, and also the function of masses corresponding to the two-loop vacuum diagram (Fig. 2) that can be expressed in terms of dilogarithms. For some special cases occurring in the Standard Model, we compared our expansion with the result of the numerical integration [11], and we found good agreement in the region above the highest threshold. The asymptotic value as $k^2 \rightarrow \infty$ is, of course, $-6\zeta(3)\pi^4/k^2$ (in the case of unit powers of denominators).

Recently some of these algorithms have been used for the renormalization group analysis of hadronic decays of a charged Higgs boson [17].

2. Three- and four-point massless diagrams

The methods considered in Section 1 can be also applied to massive diagrams with larger number of external lines. When constructing the expansion for large values of the external momentum invariants, we shall need results for the corresponding diagrams with massless internal particles. The examination of such diagrams is also important for calculation of some types of radiative corrections and for analysis of some theoretical models.

In papers [3, 5] we considered two-loop three- and four-point contributions with off-shell external momenta. We have also considered three- and four-point ladder con-

tributions with an arbitrary number of loops [4]. We used the following tools: (i) the Feynman parametric representation, (ii) the “uniqueness” conditions (see, e.g., in [18]), (iii) Mellin–Barnes contour integrals and (iv) Fourier transform to the coordinate space. We considered only scalar diagrams (corresponding to the massless ϕ^3 theory), because expressions occurring in realistic calculations can be reduced to such scalar integrals.

For the two-loop three-point diagram $C^{(2)}$ (see Fig. 4a), we obtained a simple integral

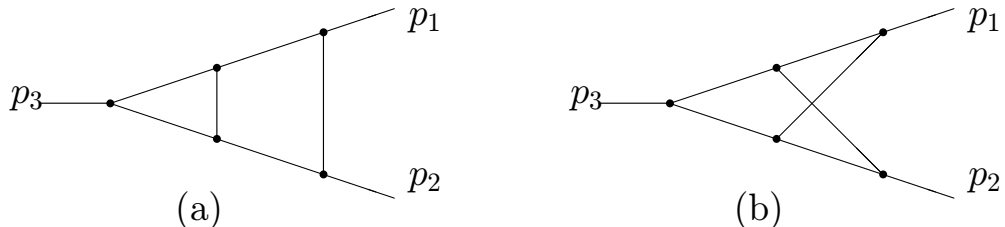


Figure 4: The planar (a) and non-planar (b) two-loop three-point diagrams

representation,

$$C^{(2)}(p_1^2, p_2^2, p_3^2) = -\frac{1}{2} \left(\frac{i\pi^2}{p_3^2} \right)^2 \int_0^1 \frac{d\xi \ln \xi}{y\xi^2 + (1-x-y)\xi + x} \left(\ln \frac{y}{x} + \ln \xi \right) \left(\ln \frac{y}{x} + 2 \ln \xi \right) \quad (3)$$

with $x \equiv p_1^2/p_3^2$, $y \equiv p_2^2/p_3^2$. The integral (3) can be easily calculated in terms of polylogarithms Li_N (see in [10]) with $N \leq 4$.

For the non-planar (“crossed”) two-loop three-point diagram $\tilde{C}^{(2)}(p_1^2, p_2^2, p_3^2)$ (see Fig. 4b), we found that

$$\tilde{C}^{(2)}(p_1^2, p_2^2, p_3^2) = \left(C^{(1)}(p_1^2, p_2^2, p_3^2) \right)^2, \quad (4)$$

where $C^{(1)}$ is a function corresponding to one-loop triangle diagram,

$$C^{(1)}(p_1^2, p_2^2, p_3^2) = -\frac{i\pi^2}{p_3^2} \int_0^1 \frac{d\xi}{y\xi^2 + (1-x-y)\xi + x} \left(\ln \frac{y}{x} + 2 \ln \xi \right). \quad (5)$$

Since the integral (5) can be evaluated in terms of dilogarithms Li_2 , the result for $\tilde{C}^{(2)}$ contains products of dilogarithms.

We examined a four-point ladder diagram $D^{(2)}$ (“double box”, see Fig. 5), and showed that the corresponding function can be reduced to $C^{(2)}$, namely,

$$D^{(2)}(k_1^2, k_2^2, k_3^2, k_4^2, s, t) = t C^{(2)}(k_1^2, k_3^2, k_2^2, k_4^2, s, t), \quad (6)$$

where $s \equiv (k_1 + k_2)^2$ and $t \equiv (k_2 + k_3)^2$ are usual Mandelstam variables. Therefore, the “double box” can also be evaluated in terms of Li_N with $N \leq 4$.

We managed to generalize some of the results for one- and two-loop diagrams to the case of arbitrary number of loops [4]. We considered L -loop three-point ladder diagrams $C^{(L)}$ (Fig. 6a), and we obtained the following generalization of the formulae (5), (3):

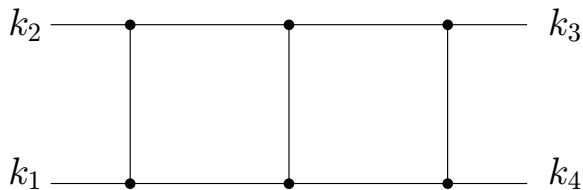


Figure 5: The “double box” diagram

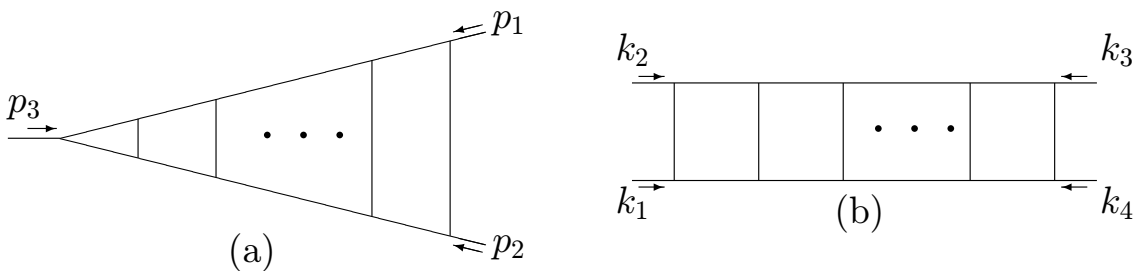


Figure 6: The three-point (a) and four-point (b) L -loop diagrams

$$\begin{aligned}
 C^{(L)}(p_1^2, p_2^2, p_3^2) &= -\frac{1}{L! (L-1)!} \left(\frac{i\pi^2}{p_3^2} \right)^L \\
 &\times \int_0^1 \frac{d\xi}{y\xi^2 + (1-x-y)\xi + x} \ln^{L-1} \xi \left(\ln \frac{y}{x} + \ln \xi \right)^{L-1} \left(\ln \frac{y}{x} + 2 \ln \xi \right). \quad (7)
 \end{aligned}$$

This integral can be calculated in terms of polylogarithms Li_N with $N \leq 2L$.

Moreover, for the four-point L -loop ladder diagram $D^{(L)}$ (shown in Fig. 6b) we found that

$$D^{(L)}(k_1^2, k_2^2, k_3^2, k_4^2, s, t) = t^{L-1} C^{(L)}(k_1^2 k_3^2, k_2^2 k_4^2, st). \quad (8)$$

This formula generalizes eq. (6) and holds also for one-loop and zero-loop (tree) cases. Thus, the four-point L -loop diagram can be also expressed in terms of Li_N with $N \leq 2L$. Note that another derivation of the formula (8) was presented in ref. [19].

We have checked the results for L -loop diagrams by reducing them to the $(L+1)$ -loop two-point functions (by means of additional integration), and we found complete agreement with known results for such two-point diagrams [20].

Acknowledgements

I would like to thank the AIHENP-93 Organizing Committee for their help, and the International Science Foundation (New York) for financial support. Research was supported, in part, by the Research Council of Norway.

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