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A simple formula for reducing Feynman diagrams to scalar integrals

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Abstract. An explicit general formula is obtained which makes it possible to reduce tensor Feynman integrals (corresponding to arbitrary one-loop N-point diagrams) to scalar integrals.

1. It is very important to develop methods and algorithms which enable one to evaluate effectively various types of Feynman diagrams. This is connected with the necessity of performing a large number of calculations in various gauge theories (QCD, electroweak model, etc.). Since in realistic calculations we are often confronted with spinor and vector particles, we are compelled to deal with integrals with tensor structures in the numerator.

A standard approach to evaluating such tensor integrals has been developed in refs. [1]–[4] (a modified procedure has been proposed in ref. [5]). It involves the following steps: (i) the tensor integral is represented as a sum of independent tensor structures (formed from the external momenta and the metric tensor) multiplied by scalar quantities; (ii) by considering various contractions, a system of linear equations is obtained from which we define these scalar quantities as integrals with scalar numerators; (iii) scalar numerators are represented in terms of denominators and we obtain a representation in terms of initial scalar integrals. Although this approach makes it possible to solve the problem in principle, nevertheless, the expressions obtained turn out to be very cumbersome when the number of independent external momenta increases (see, e.g., ref. [3]). In addition, this approach has some complications connected with the appearance of "kinematic" determinants in the expressions obtained.

In the present paper we propose another approach to this problem. It is based on some relations connecting integrals in different space-time dimensions (see below). This approach enables us to derive a simple general formula for one-loop N-point tensor integrals. It should be noted that a particular case of such a formula for a class of three-point integrals has been examined earlier in ref. [6].

2. The one-loop N-point tensor integrals we are interested in are of the following form:

$$J^{(N)}_{\mu_1\ldots\mu_M}(n;\nu_1,\ldots,\nu_N) \equiv \int \frac{q_{\mu_1}\ldots q_{\mu_M}\,\mathrm{d}^n q}{D_1^{\nu_1}\ldots D_N^{\nu_N}}\ , \tag{1}$$

where $D_j \equiv (p_j+q)^2 - m_j^2 + \mathrm{i}0$ are the massive denominators (some of the masses can be taken equal to zero). We shall consider the powers of the denominators ν_j and the space-time dimension n as arbitrary parameters. This enables us to deal with analytically- and dimensionally-regularized integrals. Sometimes we shall use the notation $\{\nu_j\} \equiv (\nu_1,\ldots,\nu_N)$. The corresponding scalar integral ($M=0$) is of the form

$$J^{(N)}(n;\nu_1,\ldots,\nu_N) \equiv \int \frac{\mathrm{d}^n q}{D_1^{\nu_1}\ldots D_N^{\nu_N}}\ . \tag{2}$$

The integrals (1) and (2) correspond to the N-point Feynman diagram of fig. 1. It should be noted that the integration momentum q is usually chosen so that one of the external momenta p_j vanishes (e.g., $p_N=0$ and $D_N = q^2 - m_N^2 + \mathrm{i}0$). For the purpose of obtaining symmetric (with respect to the indices $1,\ldots,N$) results we shall keep all the momenta p_j arbitrary (if necessary one can put $p_N=0$ in the final expressions).

Let us consider first the vector integral $J^{(N)}_{\mu}$ corresponding to the case when $M=1$ (see (1)). This integral can be obtained from (2) by differentiation with respect to the external momentum (e.g., p_1)

$$J^{(N)}_{\mu}(n;\{\nu_j\}) = -p_{1\mu} J^{(N)}(n;\{\nu_j\}) - \frac{1}{2(\nu_1-1)}\frac{\partial}{\partial p_{1\mu}} J^{(N)}(n;\{\nu_j-\delta_{j1}\})\ , \tag{3}$$

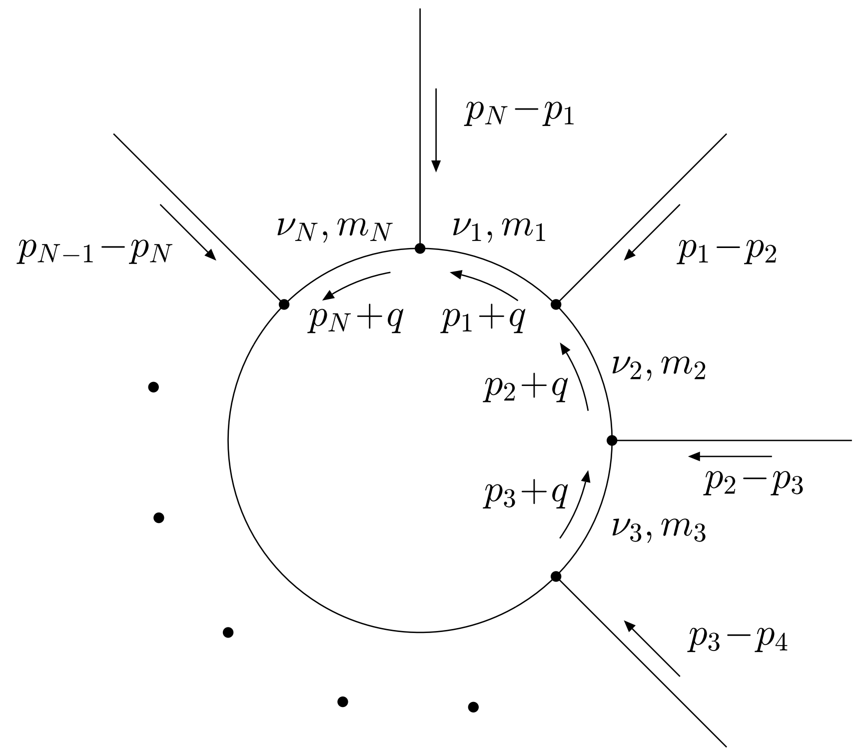

Fig. 1. The one-loop N-point diagram

where δ_{jk} is the Kronecker symbol.

Below we shall need some relations connecting different scalar integrals $J^{(N)}$. To obtain these formulae, it is convenient to use the α-representation for the integral (2). Representing all denominators $D_i^{-\nu_i}$ in the form of the integrals with respect to the α_i-parameters, integrating over the momentum q, performing a standard variables substitution ($\sum \alpha_i = \Lambda$, $\alpha_i = \Lambda \beta_i$, $\sum \beta_i = 1$) and integrating over Λ, we obtain

$$J^{(N)}(n; \nu_1, \ldots, \nu_N) = \pi^{n/2} \, \mathrm{i}^{1-n} \, \Gamma\left(\sum \nu_i - \tfrac{1}{2}n\right) \left(\prod \Gamma(\nu_i)\right)^{-1}$$
$$\times \int\limits_0^1 \ldots \int\limits_0^1 \prod \beta_i^{\nu_i - 1} \mathrm{d}\beta_i \; \delta\left(\sum \beta_i - 1\right) \left(\sum\sum_{j<l} \beta_j \beta_l (p_j - p_l)^2 - \sum \beta_i m_i^2\right)^{n/2 - \sum \nu_i} . \quad (4)$$

Here and henceforth $\sum$ and $\prod$ denote the sum and the product from 1 to N (if these limits are not written explicitly). By using the representation (4), one can easily prove two useful formulae:

$$\frac{\partial}{\partial p_{1\mu}} J^{(N)}(n; \{\nu_j\}) = 2\nu_1 \sum_{k=2}^{N} (p_1 - p_k)_\mu \, \nu_k \, \pi^{-1} \, J^{(N)}(n+2; \{\nu_j + \delta_{j1} + \delta_{jk}\}) \, , \quad (5)$$

$$J^{(N)}(n; \{\nu_j\}) = -\pi^{-1} \sum_{k=1}^{N} \nu_k \, J^{(N)}(n+2; \{\nu_j + \delta_{jk}\}) \, . \quad (6)$$

Note that the scalar integrals on the r.h.s. of (5) and (6) have the space-time dimension $(n+2)$ rather than n. The π^{-1}-factors appear from $\pi^{n/2}$ in (4). If we consider, for example, $\pi^{-n/2}\, J^{(N)}$ instead of $J^{(N)}$ such factors will not appear. Finally, combining (5) and (6) we get

$$\frac{\partial}{\partial p_{1_\mu}} J^{(N)}(n;\{\nu_j-\delta_{j1}\}) = -2(\nu_1-1)p_{1_\mu}\, J^{(N)}(n;\{\nu_j\}) -2(\nu_1-1)\sum_{k=1}^{N} p_{k_\mu}\,\nu_k\,\pi^{-1}\, J^{(N)}(n+2;\{\nu_j+\delta_{jk}\}) \,. \quad (7)$$

Inserting formula (7) into (3) yields

$$J_\mu^{(N)}(n;\nu_1,\ldots,\nu_N) = \sum_{k=1}^{N} p_{k_\mu}\,\nu_k\,\pi^{-1}\, J^{(N)}(n+2;\{\nu_j+\delta_{jk}\}) \,. \quad (8)$$

We see that the vector integral $J_\mu^{(N)}$ is expressed through the sum of the external vectors p_{k_μ} multiplied by the scalar integrals in $(n+2)$ dimensions, each coefficient of p_{k_μ} being expressed through one scalar integral only. For example, when $N=2$ and $p_2=0$ $(p_1\equiv p)$ we have

$$J_\mu^{(2)}(n;\nu_1,\nu_2) = p_\mu\,\nu_1\,\pi^{-1}\, J^{(2)}(n+2;\nu_1+1,\nu_2) \,.$$

Let us examine now the case $M=2$ of formula (1). By analogy with (3), the tensor integral $J_{\mu_1\mu_2}^{(N)}$ can be expressed through $J_{\mu_1}^{(N)}$,

$$J_{\mu_1\mu_2}^{(N)}(n;\{\nu_j\}) = -p_{1_{\mu_2}} J_{\mu_1}^{(N)}(n;\{\nu_j\}) - \frac{1}{2(\nu_1-1)}\,\frac{\partial}{\partial p_{1_{\mu_2}}} J_{\mu_1}^{(N)}(n;\{\nu_j-\delta_{j1}\}) \,. \quad (9)$$

Inserting expression (8) for $J_{\mu_1}^{(N)}$ and using (7) we obtain

$$\begin{aligned} J_{\mu_1\mu_2}^{(N)}(n;\nu_1,\ldots,\nu_N) &= -\tfrac{1}{2} g_{\mu_1\mu_2}\pi^{-1} J^{(N)}(n+2;\{\nu_j\}) \\ &+\sum_{k=1}^{N} p_{k_{\mu_1}} p_{k_{\mu_2}}\,\nu_k\,(\nu_k+1)\,\pi^{-2}\, J^{(N)}(n+4;\{\nu_j+2\delta_{jk}\}) \\ &+\sum_{k<k'}\sum (p_{k_{\mu_1}} p_{k'_{\mu_2}} + p_{k'_{\mu_1}} p_{k_{\mu_2}})\,\nu_k\,\nu_{k'}\,\pi^{-2} J^{(N)}(n+4;\{\nu_j+\delta_{jk}+\delta_{jk'}\}) \,. \end{aligned} \quad (10)$$

The scalar factor at $g_{\mu_1\mu_2}$ is expressed through the integral in $(n+2)$ dimensions, while other integrals on the r.h.s. of (10) have the dimension $n+4$. As in the case of (8), each scalar factor is expressed through one scalar integral only.

3. Examination of expressions (8) and (10) (as well as the way of obtaining these results) enables us to construct a general formula for the integrals (1) (for arbitrary values of M):

$$\begin{aligned} J_{\mu_1\ldots\mu_M}^{(N)}(n;\nu_1,\ldots,\nu_N) = &\sum_{\substack{\lambda,\kappa_1,\ldots,\kappa_N \\ 2\lambda+\Sigma\kappa_i=M}} \left(-\tfrac{1}{2}\right)^{\lambda}\, \{[g]^{\lambda}\,[p_1]^{\kappa_1}\ldots[p_N]^{\kappa_N}\}_{\mu_1\ldots\mu_M} \\ &\times(\nu_1)_{\kappa_1}\ldots(\nu_N)_{\kappa_N}\,\pi^{\lambda-M}\, J^{(N)}(n+2(M-\lambda);\nu_1+\kappa_1,\ldots,\nu_N+\kappa_N) \,, \end{aligned} \quad (11)$$

where $(\nu)_\kappa \equiv \Gamma(\nu+\kappa)/\Gamma(\nu)$ is the Pochhammer symbol. The structure

$$\{[g]^{\lambda}\,[p_1]^{\kappa_1}\ldots[p_N]^{\kappa_N}\}_{\mu_1\ldots\mu_M}$$

is the symmetric (with respect to $\mu_1, \ldots, \mu_M$) tensor combination, each term of which is constructed from λ metric tensors g, κ_1 momenta p_1, …, κ_N momenta p_N. For example,

$$\{gp_1\}_{\mu_1\mu_2\mu_3} = g_{\mu_1\mu_2}p_{1\mu_3} + g_{\mu_1\mu_3}p_{1\mu_2} + g_{\mu_2\mu_3}p_{1\mu_3} \; .$$

In formula (11) the sum extends over all possible non-negative values of $\lambda, \kappa_1, \ldots, \kappa_N$, with the restriction that the rank of the tensor structures, $(2\lambda + \sum \kappa_i)$, should be equal to M. Therefore, $\max \kappa_i = M$ and $\max \lambda = [M/2]$ (integer part of $M/2$).

We shall prove the formula (11) by induction. Let this formula be true for some value of M. Then we have, by analogy with (9), that

$$\begin{aligned} J^{(N)}_{\mu_1\ldots\mu_M\mu_{M+1}}(n;\nu_1,\ldots,\nu_N) &= -p_{1\mu_{M+1}} J^{(N)}_{\mu_1\ldots\mu_M}(n;\nu_1,\ldots,\nu_N) \\ &\quad -\frac{1}{2(\nu_1-1)}\frac{\partial}{\partial p_{1\mu_{M+1}}} J^{(N)}_{\mu_1\ldots\mu_M}(n;\nu_1-1,\ldots,\nu_N) \; . \end{aligned} \quad (12)$$

Inserting the expression (11) for $J^{(N)}_{\mu_1\ldots\mu_M}$ into (12), using formula (7) and taking into account that $(\nu_1-1)_{\kappa_1} = (\nu_1-1)(\nu_1)_{\kappa_1-1}$ we obtain

$$\begin{aligned} J^{(N)}_{\mu_1\ldots\mu_{M+1}}(n;\nu_1,\ldots,\nu_N) &= \sum_{\substack{\lambda,\kappa_1,\ldots,\kappa_N \\ 2\lambda+\Sigma\kappa_i=M}} \left(-\tfrac{1}{2}\right)^{\lambda+1} \frac{\partial}{\partial p_{1\mu_{M+1}}} \{[g]^\lambda\, [p_1]^{\kappa_1}\ldots[p_N]^{\kappa_N}\}_{\mu_1\ldots\mu_M} \\ &\times \left(\prod_{i=1}^N (\nu_i)_{\kappa_i-\delta_{i1}}\right) \pi^{\lambda-M}\; J^{(N)}(n+2(M-\lambda);\{\nu_j+\kappa_j-\delta_{j1}\}) \\ &+ \sum_{k=1}^N \sum_{\substack{\lambda,\kappa_1,\ldots,\kappa_N \\ 2\lambda+\Sigma\kappa_i=M}} \left(-\tfrac{1}{2}\right)^{\lambda} \{[g]^\lambda\, [p_1]^{\kappa_1}\ldots[p_N]^{\kappa_N}\}_{\mu_1\ldots\mu_M}\, p_{k\mu_{M+1}} \\ &\times \left(\prod_{i=1}^N (\nu_i)_{\kappa_i+\delta_{ik}}\right) \pi^{\lambda-M-1}\; J^{(N)}(n+2(M-\lambda+1);\{\nu_j+\kappa_j+\delta_{jk}\}) \; . \end{aligned} \quad (13)$$

Substituting $(\kappa_1^{(\mathrm{old})} = \kappa_1^{(\mathrm{new})}+1)$ and $(\lambda^{(\mathrm{old})} = \lambda^{(\mathrm{new})}-1)$ in the first term on the r.h.s. of (13), and $(\kappa_k^{(\mathrm{old})} = \kappa_k^{(\mathrm{new})}-1)$ in the sums of the second term, we have

$$\begin{aligned} J^{(N)}_{\mu_1\ldots\mu_{M+1}}(n;\nu_1,\ldots,\nu_N) &= \sum_{\substack{\lambda,\kappa_1,\ldots,\kappa_N \\ 2\lambda+\Sigma\kappa_i=M+1}} \left(-\tfrac{1}{2}\right)^{\lambda} \\ &\times \left(\frac{\partial}{\partial p_{1\mu_{M+1}}} \{[g]^\lambda\, [p_1]^{\kappa_1+1}[p_2]^{\kappa_2}\ldots[p_N]^{\kappa_N}\}_{\mu_1\ldots\mu_M} + \sum_{k=1}^N \left\{[g]^\lambda \prod_{i=1}^N [p_i]^{\kappa_i-\delta_{ik}}\right\}_{\mu_1\ldots\mu_M} p_{k\mu_{M+1}} \right) \\ &\times (\nu_1)_{\kappa_1}\ldots(\nu_N)_{\kappa_N}\, \pi^{\lambda-M-1}\; J^{(N)}(n+2(M-\lambda+1);\nu_1+\kappa_1,\ldots,\nu_N+\kappa_N) \; . \end{aligned} \quad (14)$$

In formula (14) it is understood that $[g]^{\lambda-1} = 0$ at $\lambda = 0$ and $[p]^{\kappa_k-1} = 0$ at $\kappa_k = 0$. One can easily see that the first term in the large parentheses on the r.h.s. of (14) produces all the tensor structures with $g_{\mu_i\mu_{M+1}}$ $(i = 1, \ldots, M)$, while the second term gives all the structures with $p_{k_{\mu_{M+1}}}$ $(k = 1, \ldots, N)$. As a result, the terms in the large parentheses correspond to $\{[g]^\lambda\, [p_1]^{\kappa_1} \ldots [p_N]^{\kappa_N}\}_{\mu_1 \ldots \mu_{M+1}}$, and we obtain formula (11) with M substituted by $(M+1)$. Thus, we proved the general formula (11).

Let us illustrate the general formula (11) by a simple example of massless two-point integrals with $p_2 = 0$ $(p_1 \equiv p)$,

$$J^{(2)}_{\mu_1 \ldots \mu_M}(n; \nu_1, \nu_2) \equiv \int \frac{q_{\mu_1} \ldots q_{\mu_M}\, \mathrm{d}^n q}{[(p+q)^2]^{\nu_1} (q^2)^{\nu_2}} \,. \tag{15}$$

Using the well-known formula for scalar massless integrals,

$$J^{(2)}(n; \nu_1, \nu_2) = \pi^{n/2}\, \mathrm{i}^{1-n}\, (p^2)^{n/2-\nu_1-\nu_2}\, \frac{\Gamma(n/2-\nu_1)\, \Gamma(n/2-\nu_2)\, \Gamma(\nu_1+\nu_2-n/2)}{\Gamma(\nu_1)\, \Gamma(\nu_2)\, \Gamma(n-\nu_1-\nu_2)} \,, \tag{16}$$

we obtain from (11) that

$$\begin{aligned} J^{(2)}_{\mu_1 \ldots \mu_M}(n; \nu_1, \nu_2) &= \pi^{n/2}\, \mathrm{i}^{1-n}\, (p^2)^{n/2-\nu_1-\nu_2}\, (-1)^M\, \left[\Gamma(\nu_1)\, \Gamma(\nu_2)\, \Gamma(n-\nu_1-\nu_2+M)\right]^{-1} \\ &\times \sum_{\lambda=0}^{[M/2]} \left(\tfrac{1}{2} p^2\right)^\lambda\, \{[g]^\lambda [p]^{M-2\lambda}\}_{\mu_1 \ldots \mu_M} \\ &\times \Gamma(n/2-\nu_1+\lambda)\, \Gamma(n/2-\nu_2+M-\lambda)\, \Gamma(\nu_1+\nu_2-n/2-\lambda). \end{aligned} \tag{17}$$

In particular, the term with $\lambda = 0$ coincides with the result of ref. [7].

4. Thus, in the present paper we obtained the simple general formula (11) for reducing one-loop N-point Feynman diagrams to scalar integrals. It should be noted that if we choose the integration momentum in (1) so that $p_N = 0$ then only the terms with $\kappa_N = 0$ survive in the formula (11). From our point of view, a closed general form of the formula (11) has some advantages over the approach [1]–[4]. In particular, we have one scalar integral at each of the independent tensor structures only, and "kinematic" determinants do not appear. In addition, the coefficients of (11) do not depend on the masses m_j. These facts are rather useful for the algorithmization of calculations.

On the other hand, the application of this formula to realistic calculations requires expressions for the corresponding scalar integrals (2) with various values of the space-time dimension n and the powers of denominators ν_i. For some simple cases such expressions are well known (see, e.g., (16). We also note that recently in refs. [8]–[10] some new general results for one-loop integrals corresponding to diagrams with various numbers of external lines have been presented. In particular, in refs. [9, 10] expressions for scalar N-point massive integrals (2) (with arbitrary values of n and ν_j) have been obtained in the form of multiple hypergeometric functions. Note that these explicit expressions satisfy the relations (5), (6) (this fact confirms that the results are self-consistent). The formula (11) enables us to apply those results to tensor integrals and to obtain expressions in the form of functions of the same type.

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