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Exact Results for Three- and Four-Point Ladder Diagrams with an Arbitrary Number of Rungs

N. I. Ussyukina[a,*] and A. I. Davydychev[a,b,†]

[a] Institute for Nuclear Physics, Moscow State University, 119899, Moscow, Russia
[b] Instituut-Lorentz, Universiteit Leiden, P.O. Box 9506, 2300 RA Leiden, The Netherlands

Abstract

Exact results for L-loop ladder graphs with three and four external lines (in the case of massless internal particles and arbitrary external momenta) are obtained in terms of polylogarithms.

*Electronic mail address: ussyuk@npimsu.msu.su
†Electronic mail address: davyd@npimsu.msu.su

1. The problem of evaluating ladder graphs in quantum field theory has been examined for a long time (see, e.g., [1, 2] and references therein). Some of such diagrams are important for calculation of radiative corrections (especially in multi-jet processes and Bhabha scattering). Note that often we are confronted with the cases when internal particles are massless (photons, gluons), or their masses can be neglected in high-energy processes.

While the high-energy asymptotic behaviour of ladder diagrams has been studied in many publications [3, 4, 5, 6] (see also [1] and references therein), exact results have been known only for some special cases. In the present paper we extend the approach presented in [7] (where the one- and two-loop diagrams have been considered) to the case of arbitrary number of loops (rungs). We shall examine three- and four-point ladder diagrams with massless internal particles and arbitrary external momenta (formally, such diagrams correspond to massless ϕ^3 theory).

The remainder of the paper is organized as follows. Section 2 contains necessary information about one-loop triangle and box diagrams. In Section 3 we examine the L-loop three-point ladder diagram, while in Section 4 the four-point case is considered. Section 5 (conclusion) discusses the main results.

2. In this section we shall briefly remind some useful properties of one-loop triangle and box diagrams shown in Fig. 1(a,b). We shall use these formulae below,

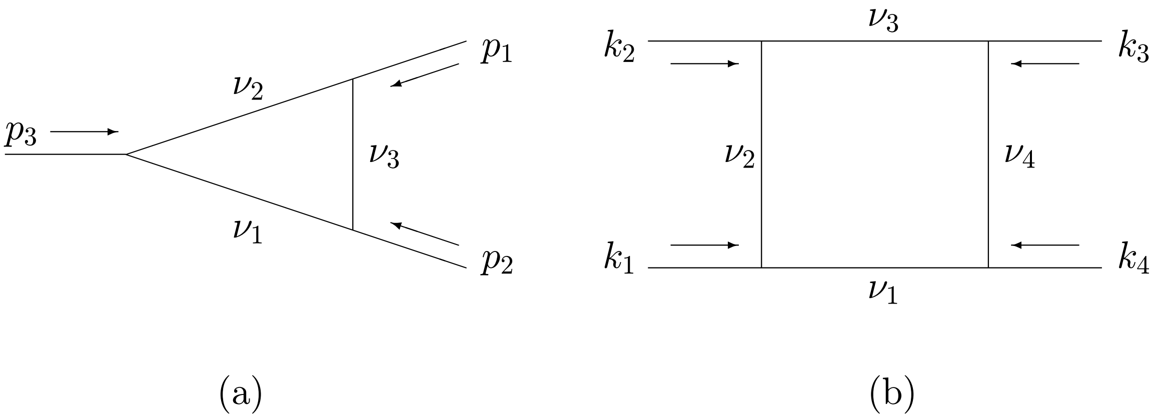

Figure 1:

when evaluating L-loop diagrams. More detailed information can be found, e.g., in [7].

Definition of the three-point Feynman integral (see Fig. 1a):

$$J_3(n;\ \nu_1,\nu_2,\nu_3|\ p_1^2,p_2^2,p_3^2) \equiv \int \frac{d^n r}{((p_2-r)^2)^{\nu_1}((p_1+r)^2)^{\nu_2}(r^2)^{\nu_3}}, \tag{1}$$

where n is the space-time dimension. Here and below the usual "causal" prescription in pseudo-Euclidean momentum space is understood, $(p^2)^{-\nu} \leftrightarrow (p^2+i0)^{-\nu}$. In some cases we shall omit the momentum arguments p_1^2, p_2^2, p_3^2 in J_3.

Double Mellin–Barnes contour integral representation [8, 9]:

$$J_3(n;\ \nu_1,\nu_2,\nu_3) = \frac{\pi^{n/2}\ i^{1-n}\ (p_3^2)^{n/2-\Sigma\nu_i}}{\Gamma\left(n-\sum\nu_i\right)\ \prod\Gamma(\nu_i)}\ \frac{1}{(2\pi i)^2}\int\limits_{-i\infty}^{i\infty}\int\limits_{-i\infty}^{i\infty} du\ dv\ x^u\ y^v\ \Gamma(-u)\Gamma(-v)$$

$$\times\Gamma(n/2-\nu_2-\nu_3-u)\Gamma(n/2-\nu_1-\nu_3-v)\Gamma(\nu_3+u+v)\Gamma(\textstyle\sum\nu_i-n/2+u+v), \qquad (2)$$

where dimensionless variables are defined by

$$x \equiv \frac{p_1^2}{p_3^2} \quad , \quad y \equiv \frac{p_2^2}{p_3^2}\ . \qquad (3)$$

In formula (2) the integration contours should separate the “right” and “left” series of poles of gamma functions in the integrand (see, e.g., [10]).

The “uniqueness” conditions (see, e.g., in [11, 12, 13, 14, 7]):

$$J_3(n;\ \nu_1,\nu_2,\nu_3)\Big|_{\Sigma\nu_i=n} = \pi^{n/2}\ i^{1-n}\prod_{i=1}^{3}\frac{\Gamma(n/2-\nu_i)}{\Gamma(\nu_i)}\ (p_i^2)^{\nu_i-n/2}, \qquad (4)$$

$$\Big\{\ \nu_1 J_3(n;\ \nu_1+1,\nu_2,\nu_3)+\nu_2 J_3(n;\ \nu_1,\nu_2+1,\nu_3)+\nu_3 J_3(n;\ \nu_1,\nu_2,\nu_3+1)\Big\}\Big|_{\Sigma\nu_i=n-2}$$

$$= \pi^{n/2}\ i^{1-n}\prod_{i=1}^{3}\frac{\Gamma(n/2-\nu_i-1)}{\Gamma(\nu_i)}\ (p_i^2)^{\nu_i-n/2+1}. \qquad (5)$$

Parametric representation for $n=4,\ \nu_1=1-\delta,\ \nu_2=1+\delta,\ \nu_3=1:$

$$J_3(4;1-\delta,1+\delta,1) = \frac{i\pi^2}{p_3^2}\ \frac{1}{\delta}\int\limits_0^1 d\xi\ \frac{\xi^{-\delta}-(\xi y/x)^{\delta}}{y\xi^2+(1-x-y)\xi+x}, \qquad (6)$$

where x and y are defined by (3). The denominator of (6) can be written as

$$p_3^2\ (y\xi^2+(1-x-y)\xi+x) = (p_1+\xi p_2)^2. \qquad (7)$$

Definition of the four-point Feynman integral (see Fig. 1b):

$$J_4(n;\ \nu_1,\nu_2,\nu_3,\nu_4|k_1^2,k_2^2,k_3^2,k_4^2,s,t) \equiv \int\frac{d^n r}{((k_4-r)^2)^{\nu_1}((k_2+k_3+r)^2)^{\nu_2}((k_3+r)^2)^{\nu_3}(r^2)^{\nu_4}}, \qquad (8)$$

where the independent Mandelstam variables are $s\equiv(k_1+k_2)^2\ ,\ t\equiv(k_2+k_3)^2$.

“Pairing” of the arguments in the case $n=4,\ \nu_1+\nu_2+\nu_3=3,\ \nu_4=1:$

$$J_4(4;\ 1-v,1+u+v,1-u,1|k_1^2,k_2^2,k_3^2,k_4^2,s,t)$$

$$= \frac{s^{u+v}}{(k_1^2)^u(k_2^2)^v}\ J_3(4;1-v,1-u,1|k_1^2k_3^2,k_2^2k_4^2,st). \qquad (9)$$

In the four-point case dimensionless variables are given by

$$X \equiv \frac{k_1^2k_3^2}{st}, \quad Y \equiv \frac{k_2^2k_4^2}{st}. \qquad (10)$$

The main steps of the proof of formula (9) have been presented in [7]. They involve (i) Feynman parametric representation, (ii) Mellin–Barnes representation and (iii) the "uniqueness" condition (4).

3. In this section we shall deal with the L-loop case of three-point ladder diagram shown in Fig. 2 (where the arrangement of momenta is also indicated). The

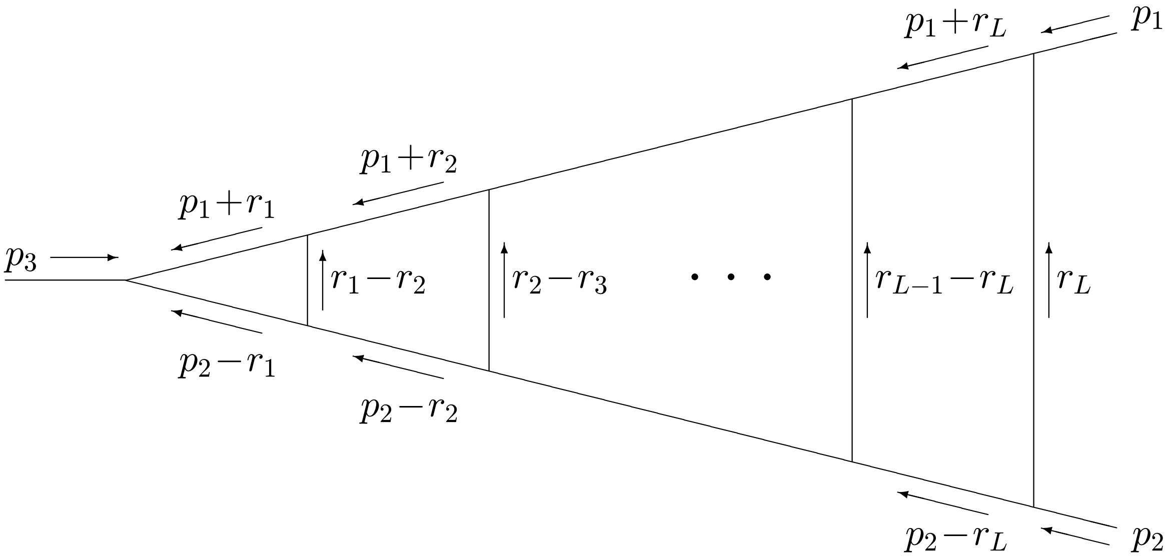

Figure 2:

corresponding Feynman integral is

$$C^{(L)}(p_1^2,p_2^2,p_3^2) \equiv \int \ldots \int \frac{d^4r_1 \ldots d^4r_L}{(r_1-r_2)^2(r_2-r_3^2)^2 \ldots (r_{L-1}-r_L)^2 r_L^2} \left\{ \prod_{i=1}^{L} (p_1+r_i)^2(p_2-r_i)^2 \right\}^{-1} . \tag{11}$$

Note that $C^{(1)}(p_1^2,p_2^2,p_3^2) \equiv J_3(4;1,1,1|p_1^2,p_2^2,p_3^2)$, where J_3 is defined by (1).

It is convenient to introduce dimensionless functions $\Phi^{(L)}$ by

$$C^{(L)}(p_1^2,p_2^2,p_3^2) = \left(\frac{i\pi^2}{p_3^2} \right)^L \Phi^{(L)}(x,y), \tag{12}$$

with x and y defined by (3). For one- and two-loop cases, simple integral representations have been derived in [7]:

$$\Phi^{(1)}(x,y) = -\int_0^1 \frac{d\xi}{y\xi^2+(1-x-y)\xi+x} \left(\ln\frac{y}{x} + 2\ln\xi \right), \tag{13}$$

$$\Phi^{(2)}(x,y) = -\frac{1}{2}\int_0^1 \frac{d\xi}{y\xi^2+(1-x-y)\xi+x} \ln\xi \left(\ln\frac{y}{x} + \ln\xi \right) \left(\ln\frac{y}{x} + 2\ln\xi \right). \tag{14}$$

By decomposition of the denominator,

$$\frac{1}{y\xi^2 + (1-x-y)\xi + x} = \frac{1}{\lambda}\left(\frac{1}{\xi+\rho x} - \frac{1}{\xi+(\rho y)^{-1}}\right), \tag{15}$$

where

$$\lambda(x,y) \equiv \sqrt{(1-x-y)^2 - 4xy} \quad , \quad \rho(x,y) \equiv \frac{2}{1-x-y+\lambda}, \tag{16}$$

these integrals can be evaluated in terms of polylogarithms (see, e.g., [15]),

$$\mathrm{Li}_N\left(z\right) = \frac{(-1)^N}{(N-1)!}\int_0^1 d\xi \, \frac{\ln^{N-1}\xi}{\xi - z^{-1}}. \tag{17}$$

The results are [7] :

$$\Phi^{(1)}(x,y) = \frac{1}{\lambda}\left\{ 2\left(\mathrm{Li}_2\left(-\rho x\right) + \mathrm{Li}_2\left(-\rho y\right)\right) + \ln\frac{y}{x}\ln\frac{1+\rho y}{1+\rho x} + \ln(\rho x)\ln(\rho y) + \frac{\pi^2}{3}\right\}, \tag{18}$$

$$\begin{aligned}\Phi^{(2)}(x,y) = \frac{1}{\lambda}\Big\{&6\left(\mathrm{Li}_4\left(-\rho x\right) + \mathrm{Li}_4\left(-\rho y\right)\right) + 3\ln\frac{y}{x}\left(\mathrm{Li}_3\left(-\rho x\right) - \mathrm{Li}_3\left(-\rho y\right)\right) \\ &+\frac{1}{2}\ln^2\frac{y}{x}\left(\mathrm{Li}_2\left(-\rho x\right) + \mathrm{Li}_2\left(-\rho y\right)\right) + \frac{1}{4}\ln^2(\rho x)\ln^2(\rho y) \\ &+\frac{\pi^2}{2}\ln(\rho x)\ln(\rho y) + \frac{\pi^2}{12}\ln^2\frac{y}{x} + \frac{7\pi^4}{60}\Big\},\end{aligned} \tag{19}$$

Note that for negative values of x and y we should take into account the $(+i0)$-prescription for denominators of (11). This requires to make the following substitutions in logarithmic terms of (18) and (19) (with $\sigma \equiv \mathrm{sgn}\; p_3^2$):

$$\ln(\rho x) \to \ln(-\rho x) + i\pi\sigma \; ; \qquad \ln(\rho y) \to \ln(-\rho y) + i\pi\sigma \; , \tag{20}$$

When evaluating two-loop diagram in the paper [7], we used special analytic regularization and uniqueness conditions (4) and (5). In such a way, the result was reduced to the functions (6). If we continue to apply this procedure (with suitable analytic regularization being the same as in [13]), we can generalize the results (13) and (14) to the L-loop case:

$$\begin{aligned}\Phi^{(L)}(x,y) = -\frac{1}{L!\,(L-1)!}\int_0^1 &\frac{d\xi}{y\xi^2 + (1-x-y)\xi + x} \\ &\times \ln^{L-1}\xi\left(\ln\frac{y}{x} + \ln\xi\right)^{L-1}\left(\ln\frac{y}{x} + 2\ln\xi\right).\end{aligned} \tag{21}$$

In the present paper we shall prove the representation (21) by induction.

Suppose, the formula (21) holds for some value of L (for $L = 1$ and $L = 2$, it coincides with the results (13) and (14), respectively). Transition from l loops to $(L+1)$ yields an additional integration (see Fig. 2):

$$\Phi^{(L+1)}(x,y) = \frac{p_3^2}{i\pi^2}\int \frac{d^4r}{r^2(p_1+r)^2(p_2-r)^2}\,\Phi^{(L)}\left(\frac{(p_1+r)^2}{p_3^2}\,,\,\frac{(p_2-r)^2}{p_3^2}\right). \tag{22}$$

If we insert the representation (21) for $\Phi^{(L)}$ into (22), we get the forth denominator, $(p_1+\xi p_2+(1-\xi)r)^2$. The logarithms $\ln(p_2-r)^2/(p_1+r)^2$ occurring in the numerator, can be transformed into derivatives with respect to the powers of denominators by

$$\frac{1}{(p_1+r)^2(p_2-r)^2}\ln^k\left(\frac{(p_2-r)^2}{(p_1+r)^2}\right)=\left(\frac{\partial^k}{\partial\delta^k}\frac{1}{((p_1+r)^2)^{1+\delta}\,((p_2-r)^2)^{1-\delta}}\right)\Bigg|_{\delta=0}. \quad (23)$$

The remaining box integral over the momentum r is

$$J_4\left(4;1-\delta,1,1+\delta,1|\ (1-\xi)^{-2}p_3^2,\ \xi^2(1-\xi)^{-2}p_3^2,\ p_1^2,\ p_2^2,\ p_3^2,\ (1-\xi)^{-2}(p_1+\xi p_2)^2\right). \quad (24)$$

By use of the "pairing" property (9), this integral can be reduced to the three-point function (1),

$$\frac{1}{p_3^2}\,(1-\xi)^2\,\xi^{-2\delta}\,J_3\left(4;1-\delta,1+\delta,1|\ p_1^2,\ \xi^2p_2^2,\ (p_1+\xi p_2)^2\right). \quad (25)$$

Using the representation (6) and substituting the variables (to restore a usual form of the denominator (7)), we find

$$\Phi^{(L+1)}(x,y)=-\frac{1}{L!\,(L-1)!}\int\limits_0^1\frac{d\xi}{y\xi^2+(1-x-y)\xi+x}\int\limits_\xi^1\frac{d\eta}{\eta}$$

$$\times\left\{\ln^{L-1}\eta\left(\frac{\partial}{\partial\delta}+\ln\eta\right)^L+\ln^L\eta\left(\frac{\partial}{\partial\delta}+\ln\eta\right)^{L-1}\right\}\left(\frac{1}{\delta}\,\eta^{-\delta}\left(\xi^{-\delta}-(\xi y/x)^\delta\right)\right)\Bigg|_{\delta=0}. \quad (26)$$

Then, after using the obvious "commutation" rule

$$\left(\frac{\partial}{\partial\delta}+\ln\eta\right)^L\eta^{-\delta}\ f(\delta)=\eta^{-\delta}\ \frac{\partial^L}{\partial\delta^L}\ f(\delta), \quad (27)$$

we can put $\eta^{-\delta}=1$ and integrate over η. Finally, evaluating the derivatives with respect to δ, we arrive at the same result as (21) with L substituted by $(L+1)$, which was to be proved.

The integral (21) can be also evaluated in terms of polylogarithms (17). Using the decomposition of denominator (15), we find

$$\Phi^{(L)}(x,y)=-\frac{1}{L!\,\lambda}\sum_{j=L}^{2L}\frac{(-1)^j\,j!\,\ln^{2L-j}(y/x)}{(j-L)!\,(2L-j)!}\left\{\mathrm{Li}_j\left(-\frac{1}{\rho x}\right)-\mathrm{Li}_j\left(-\rho y\right)\right\}, \quad (28)$$

with λ and ρ defined by (16). Note that the highest order of polylogarithms is $2L$ (for L-loop diagram). The expression (28), however, is not explicitly symmetric with respect to x and y. To restore this "hidden" symmetry, we can transform $\mathrm{Li}_j(-1/\rho x)$ to the inverse variable (see [15]), and we get

$$\Phi^{(L)}(x,y)=\frac{1}{\lambda}\left\{\frac{1}{L!}\sum_{j=L}^{2L}\frac{j!\,\ln^{2L-j}(y/x)}{(j-L)!\,(2L-j)!}\left(\mathrm{Li}_j\left(-\rho x\right)+(-1)^j\,\mathrm{Li}_j\left(-\rho y\right)\right)\right.$$

$$\left.+2\sum_{\substack{k=0\\ k+l-\text{ even}}}^{L}\sum_{l=0}^{L}\frac{(k+l)!\,(1-2^{1-k-l})}{k!\,l!\,(L-k)!\,(L-l)!}\,\zeta(k+l)\ln^{L-k}(\rho x)\,\ln^{L-l}(\rho y)\right\}, \quad (29)$$

where the coefficients of the second sum on the r.h.s. are expressed in terms of the Riemann's ζ function of even arguments (for example, $\zeta(0) = -1/2,\ \zeta(2) = \pi^2/6,\ \zeta(4) = \pi^4/90,\ \zeta(6) = \pi^6/945$, etc.). In general, rational factors at the powers of π can be related to Bernoulli numbers (see, e.g., in [15]). Note that the leading asymptotic behavior as $x \to 0,\ y \to 0$ is given by the term with $k = l = 0$ of double sum in braces on the r.h.s. of (29),

$$\Phi^{(L)}(x, y)\Big|_{x\to 0,\ y\to 0} \sim \ln^L(\rho x)\ \ln^L(\rho y), \tag{30}$$

while the "polylogarithmic" terms in the first sum of (29) vanish in this limit (for $L \geq 1$).

It is easy to check that for $L = 1$ and $L = 2$ the expression (29) coincides with the results (18) and (19), respectively (we should remember that $\mathrm{Li}_1(z) = -\ln(1-z)$). Moreover, if we consider the case $L = 0$ (remembering that $\mathrm{Li}_0(z) = z/(1-z)$), we obtain, from (29), a correct "tree" vertex $\Phi^{(0)} = 1$.

Another way to check the result (21) is to apply it to calculation of the L-loop propagator-type ladder diagram,

$$B^{(L)}(k^2) = \int \frac{d^4 r}{r^2\ (k+r)^2}\ C^{(L-1)}(r^2,\ (k+r)^2,\ k^2). \tag{31}$$

Inserting the expression (21) for the three-point function and using the formulae (23), (6) and (27), we arrive at the well-known result (see [13, 16]):

$$B^{(L)}(k^2) = \frac{(i\pi^2)^L}{(k^2)^{L-1}}\ \frac{(2L)!}{(L!)^2}\ \zeta(2L-1). \tag{32}$$

4. Let us consider now the L-loop four-point ladder diagram ("multiple box") shown in Fig. 3 (the number of rungs is $L+1$). The corresponding Feynman integral

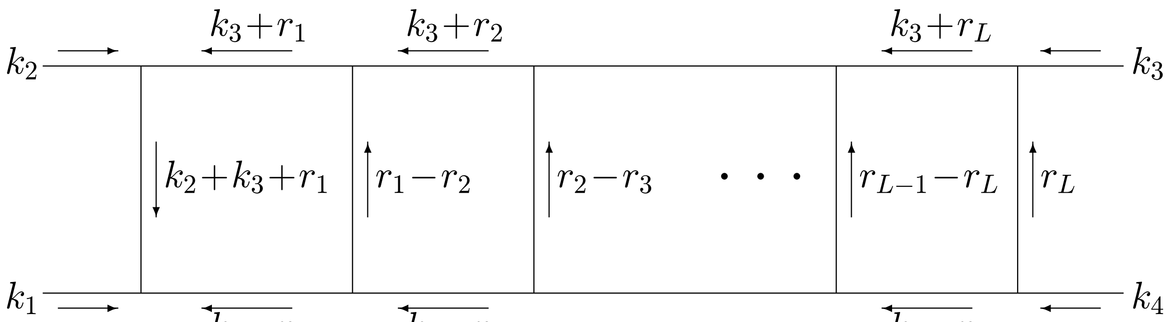

Figure 3:

is defined by

$$D^{(L)}(k_1^2, k_2^2, k_3^2, k_4^2, s, t) \equiv \int \ldots \int \frac{d^4 r_1 \ldots d^4 r_L}{(k_2+k_3+r_1)^2 (r_1-r_2)^2 (r_2-r_3)^2 \ldots (r_{L-1}-r_L)^2 r_L^2}$$

$$\times \left\{ \prod_{i=1}^{L} (k_3 + r_i)^2 (k_4 - r_i)^2 \right\}^{-1} , \quad (33)$$

where, as usual, $s = (k_1 + k_2)^2, \ t = (k_2 + k_3)^2$.

In the paper [7] it was shown that, for $L = 1$ and $L = 2$, the functions (33) can be reduced to the corresponding three-point functions (11) by

$$D^{(1)}(k_1^2, k_2^2, k_3^2, k_4^2, s, t) = C^{(1)}(k_1^2 k_3^2, k_2^2 k_4^2, st), \quad (34)$$

$$D^{(2)}(k_1^2, k_2^2, k_3^2, k_4^2, s, t) = t \, C^{(2)}(k_1^2 k_3^2, k_2^2 k_4^2, st) \quad (35)$$

(note that the formula (34) follows from (9) if we put $u = v = 0$). Moreover, in the "0-loop" (tree) case we get $D^{(0)} = 1/t = t^{-1} C^{(0)}$. Looking at these formulae, we can suppose that analogous "pairing" property is valid also for the L-loop case, namely:

$$D^{(L)}(k_1^2, k_2^2, k_3^2, k_4^2, s, t) = t^{L-1} \, C^{(L)}(k_1^2 k_3^2, k_2^2 k_4^2, st). \quad (36)$$

We shall prove the formula (36) by induction. Let us suppose that (36) holds for some value of L (we know that it is true for $L = 0, 1, 2$). It is convenient to introduce Mellin–Barnes representation,

$$C^{(L)}(p_1^2, p_2^2, p_3^2) = \left(\frac{i\pi^2}{p_3^2} \right)^L \frac{1}{(2\pi i)^2} \int\limits_{-i\infty}^{i\infty} \int\limits_{-i\infty}^{i\infty} du \, dv \left(\frac{p_1^2}{p_3^2} \right)^u \left(\frac{p_2^2}{p_3^2} \right)^v \mathcal{M}^{(L)}(u, v). \quad (37)$$

In fact, $\mathcal{M}^{(L)}(u, v)$ is the Mellin-transformed image of the function $\Phi^{(L)}(x, y)$ (12). For example, for $L = 1$ we find, from (2) (see also in [17]), that

$$\mathcal{M}^{(1)}(u, v) = \Gamma^2(-u) \, \Gamma^2(-v) \, \Gamma^2(1 + u + v) \quad (38)$$

Considering the $(L+1)$-loop function (22) and using (2), it is easy to obtain that

$$\mathcal{M}^{(L+1)}(u, v) = \Gamma(-u) \, \Gamma(-v) \, \Gamma(1 + u + v) \, \frac{1}{(2\pi i)^2} \int\limits_{-i\infty}^{i\infty} \int\limits_{-i\infty}^{i\infty} du' \, dv'$$
$$\times \frac{\Gamma(u' - u) \, \Gamma(v' - v) \, \Gamma(1 + u + v - u' - v')}{\Gamma(1 - u') \, \Gamma(1 - v') \, \Gamma(1 + u' + v')} \, \mathcal{M}^{(L)}(u', v'). \quad (39)$$

Let us consider then the $(L+1)$-loop four-point function,

$$D^{(L+1)}(k_1^2, k_2^2, k_3^2, k_4^2, s, t) = \int \frac{d^4 r}{r^2 \, (k_3 + r)^2 \, (k_4 - r)^2}$$
$$\times D^{(L)}(k_1^2, k_2^2, (k_3 + r)^2, (k_4 - r)^2, s, (k_2 + k_3 + r)^2). \quad (40)$$

By supposition, in the L-loop case $D^{(L)}$ can be reduced to $C^{(L)}$ (36). Using the Mellin–Barnes representation for $C^{(L)}$ (37), we find that

$$D^{(L+1)}(k_1^2, k_2^2, k_3^2, k_4^2, s, t) = \left(\frac{i\pi^2}{s} \right)^L \frac{1}{(2\pi i)^2} \int\limits_{-i\infty}^{i\infty} \int\limits_{-i\infty}^{i\infty} du \, dv \, \mathcal{M}^{(L)}(u, v)$$
$$\times J_4(4; 1 - v, 1 + u + v, 1 - u, 1 | k_1^2, k_2^2, k_3^2, k_4^2, s, t). \quad (41)$$

The box integral J_4 has exactly the same powers of denominators as it is needed to apply the "pairing" property (9). Using (9) and (2), we arrive at the representation of $D^{(L+1)}$ that coincides with that of $C^{(L+1)}$ (see (37)–(39)) if we substitute $p_1^2 \to k_1^2 k_3^2$, $p_2^2 \to k_2^2 k_4^2$, $p_3^2 \to st$ and multiply the result by t^L. Thus, we proved that the formula (36) is valid also for the $(L+1)$-loop case, q.e.d.

So, the L-loop ladder four-point function is expessed in terms of three-point function (12),

$$D^{(L)}(k_1^2, k_2^2, k_3^2, k_4^2, s, t) = \frac{(i\pi^2)^L}{s^L\; t}\, \Phi^{(L)}(X, Y), \tag{42}$$

where X and Y are defined by (10). Thus, the function $D^{(L)}$ (33) can be represented through simple integral representation (21), or in terms of polylogarithms (29). In a physically interesting case (when s and k_i^2 are positive while t is negative), X and Y are negative, and one needs to use a prescription of the type of (20). It should be also noted that the s-channel of the diagram in Fig. 3 corresponds to a "horizontal" ladder with initial particles momenta k_1 and k_2, while the t-channel corresponds to a "vertical" ladder with initial momenta k_2 and k_3.

5. In the present paper we considered three- and four-point ladder diagrams (Fig. 2 and Fig. 3) with arbitrary number of loops (rungs). It was shown that the L-loop four-point diagram $D^{(L)}$ (33) can be reduced to the three-point function $C^{(L)}$ (11). Both of them can be expressed via dimensionless function $\Phi^{(L)}$ (see (12) and (42)). For the latter we obtained a simple integral representation (21) that can be evaluated in terms of polylogarithms (29). The representation (21) is convenient when we use the corresponding functions as "blocks" in multiloop calculations, because the denominator (7) can be associated with a propagator.

If we use the result (29) we can see that the leading logarithmic terms (30) yield a correct asymptotic behaviour as $x \to 0$, $y \to 0$ (while the polylogarithms are regular functions in this limit). For four-point diagrams, the limit $X \to 0$, $Y \to 0$ (see (10)) corresponds to high values of s or t. On the other hand, the same situation occurs when some of the external particles are on shell (for example, when we consider corrections to multi-gluon jet processes). The appearing infrared (on-shell) singularities can be also parametrized by $1/(n-4)$ poles in dimensional regularization [18] (see, e.g., two-loop three-point examples in [19]). In our (four-dimensional) approach we can put vanishing external momenta squared equal to μ^2, and the singularities will appear as the powers of $\ln \mu$. There is a problem, however, how to relate non-leading $1/(n-4)$ and $\ln \mu$ singularities.

A simple form of the representation (21) makes it possible to consider the infinite sum of ladder diagrams with all numbers of rungs. As a result, we obtain in the numerator of the integrand a Bessel-type function, and we can analyse properties of this representation. This is one of seldom examples of explicit summability of perturbation theory series.

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References

[1] R.J.Eden, P.V.Landshoff, D.I.Olive and J.C.Polkinghorne, *The analytic S-matrix* (University Press, Cambridge, 1966).

[2] P.D.B.Collins, *An introduction to Regge theory and high energy physics* (University Press, Cambridge, 1977).

[3] P.Osland and T.T.Wu, *Nucl.Phys.* B288 (1987) 77; 95.

[4] M.C.Bergère and L.Szymanowski, *Phys.Lett.* B237 (1990) 503; *Nucl.Phys.* B350 (1991) 82.

[5] R.Gastmans and W.Troost, *Phys.Lett.* B249 (1990) 523;
R.Gastmans, W.Troost and T.T.Wu, *Nucl.Phys.* B365 (1991) 404.

[6] K.S.Bjørkevoll, G.Fäldt and P.Osland, Bergen Univ. reports 1991-07, 1992-01, 1992-05 (Bergen, 1991–1992); *Nucl.Phys.* B, to appear.

[7] N.I.Ussyukina and A.I.Davydychev, Preprint NPI MSU 92-38/287 (Moscow, 1992); *Phys.Lett.* B, to appear.

[8] E.E.Boos and A.I.Davydychev, *Teor.Mat.Fiz.* 89 (1991) 56.

[9] A.I.Davydychev, *J.Phys.* A25 (1992) 5587.

[10] W.N.Bailey, *Generalized hypergeometric series* (University Press, Cambridge, 1935).

[11] M.D'Eramo, L.Peliti and G.Parisi, *Lett. Nuovo Cim.* 2 (1971) 878;
A.N.Vassiliev, Yu.M.Pis'mak and Yu.R.Honkonen, *Teor.Mat.Fiz.* 47 (1981) 291.

[12] N.I.Ussyukina, *Teor.Mat.Fiz.* 54 (1983) 124.

[13] V.V.Belokurov and N.I.Ussyukina, *J.Phys.* A16 (1983) 2811.

[14] D.I.Kazakov, *Teor.Mat.Fiz.* 58 (1984) 345.

[15] L.Lewin, *Polylogarithms and associated functions* (North Holland, 1981).

[16] D.J.Broadhurst, *Phys.Lett.* B164 (1985) 356.

[17] N.I.Ussyukina, *Teor.Mat.Fiz.* 22 (1975) 300.

[18] G.'tHooft and M.Veltman, *Nucl.Phys.* B44 (1972) 189;
C.G.Bollini and J.J.Giambiagi, *Nuovo Cim.* 12B (1972) 20.

[19] R.J.Gonsalves, *Phys.Rev.* D28 (1983) 1542;
W.L. van Neerven, *Nucl.Phys.* B268 (1986) 453;
G.Kramer and B.Lampe, *J.Math.Phys.* 28 (1987) 945.