Loop calculations in QCD with massive quarks*

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Abstract

A method of evaluating massive Feynman loop diagrams is described. It is based on the representation of massive denominators in the form of Mellin–Barnes contour integrals. The method is illustrated by the results for some classes of one- and two-loop Feynman integrals occurring in QCD calculations. A procedure of combined application of this method and the integration-by-parts technique to more complicated two-loop massive diagrams is described.

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1. Many current problems of elementary particles theory require to develop methods and algorithms of exact evaluation of various classes of Feynman diagrams including both massless and massive particles. We can mention, for example, such problems as calculation of radiative corrections to cross sections and decay widths, QCD sum rules analysis of various quantities, some renormalization group calculations, examination of behaviours of Green functions, etc.

It is often necessary to evaluate Feynman diagrams with massive denominators (e.g., when we consider QCD with massive quarks). However, the progress in massive loop calculations is not so essential as in massless calculations. Some examples of massive calculations see, e.g., in refs. [1]–[6].

In this paper we shall briefly describe the main ideas of the method of evaluating Feynman diagrams [7]–[9] (including a new technique of reducing Feynman amplitudes to scalar integrals [10]). We shall also illustrate this method via some examples of one- and two-loop calculations.

The considered method [7]–[9] is based on the representation of massive denominators in the form of Mellin–Barnes contour integrals,

$$\frac{1}{(q^2-m^2+\mathrm{i}0)^{\nu}} = \frac{1}{\Gamma(\nu)}\,\frac{1}{2\pi\mathrm{i}}\int\limits_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}s \frac{(-m^2)^s}{(q^2+\mathrm{i}0)^{\nu+s}}\;\Gamma(-s)\;\Gamma(\nu+s)\;, \tag{1}$$

where q is the momentum of the corresponding line, m is the mass (the masses can differ for different lines), and ν is the power of the denominator (the index of the line). Imaginary infinitesimals (+i0) define an usual "causal" way of dealing with singularities in the pseudo-Euclidean space. Below we shall imply that all sqared momenta in denominators have such "additions". In formula (1) the integration contour in the complex s-plane separates "left" series of poles of Γ-functions in the integrand ($\Gamma(\nu+s)$) from "right" poles ($\Gamma(-s)$). Below we shall understand all such contour integrals in this sense. To evaluate the integrals of the type (1) one can use the resudue theorem, closing the contour in the right or in the left half-plane of the variable s, depending on the value of of $|m^2/q^2|$.

Thus, the algorithm of evaluation of a massive Feynman integral is the following: (i) all massive denominators are represented in the form of Mellin–Barnes integrals (1), (ii) the obtained massless integral with "shifted" powers of denominators is evaluated, and (iii) we evaluate the remaining contour integrals. If some of the masses are equal or there are some additional conditions on external momenta, we can reduce the number of contour integrals. If the integral is evaluated for arbitrary values of the space-time dimension and of the powers of denominators then, as a rule, the obtained general results can be represented in the form of hypergeometric functions (these results can be used in the framework of both dimensional and analytical regularization schemes). Some examples of such results will be presented in sections 2 and 3. It should be also noted that in some cases it is convenient to apply the considered method in combination with the integration-by-parts technique [11, 12] (see in section 4).

If we have an integral with tensor structure in the numerator (formed from the integration momentum), we can reduce it to scalar integrals (for example, by using the standard technique [13]). In our opinion, in some cases it is more convenient to use another algorithm of reducing tensor amplitudes to scalar integrals [10]. In ref. [10] a simple general formula for arbitrary one-loop diagrams has been obtained, which represents the tensor integral as a sum of independent tensor combinations (constructed from the external momenta and the metric tensor) multiplied by scalar integrals in higher space-time dimensions. Below we shall consider only the scalar integrals.

2. In this section we shall present some results for one-loop integrals. We shall not repeat technical details of calculations which can be found in refs. [7]–[9].

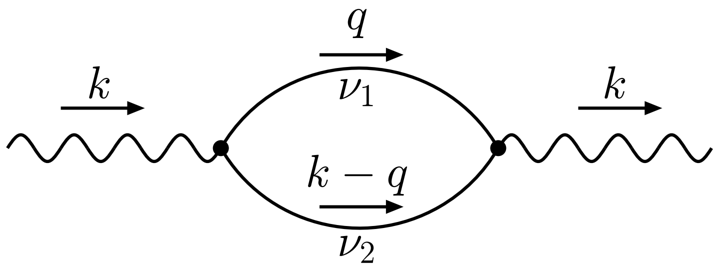


Fig. 1

Let us consider one-loop propagator-type integrals (see Fig. 1):

$$J(\nu_1,\nu_2;m) \equiv \int \frac{\mathrm{d}^n q}{(q^2-m^2)^{\nu_1}\,((k-q)^2-m^2)^{\nu_2}}, \tag{2}$$

where $n=4-2\varepsilon$ is the space-time dimension (in the framework of dimensional regularization [14]) and k is the external momentum. By using representation (1) we obtain the following contour integral:

$$J(\nu_1,\nu_2;m) = \pi^{n/2}\,\mathrm{i}^{1-n}\,(-m^2)^{n/2-\nu_1-\nu_2}\,[\Gamma(\nu_1)\Gamma(\nu_2)]^{-1} \times \frac{1}{2\pi\mathrm{i}}\int\limits_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\mathrm{d}u\left(-\frac{k^2}{m^2}\right)^u \Gamma(-u)\frac{\Gamma(\nu_1+u)\Gamma(\nu_2+u)\Gamma(\nu_1+\nu_2-n/2+u)}{\Gamma(\nu_1+\nu_2+2u)}. \tag{3}$$

If we close the integration contour to the right we find

$$J(\nu_1,\nu_2;m) = \pi^{n/2}\,\mathrm{i}^{1-n}\,(-m^2)^{n/2-\nu_1-\nu_2} \times\frac{\Gamma(\nu_1+\nu_2-n/2)}{\Gamma(\nu_1+\nu_2)}\;{}_3F_2\left(\left.\begin{array}{c}\nu_1,\ \nu_2,\ \nu_1+\nu_2-n/2\\ (\nu_1+\nu_2)/2,\ (\nu_1+\nu_2+1)/2\end{array}\right|\frac{k^2}{4m^2}\right), \tag{4}$$

where the function ${}_3F_2$ corresponds to standard notation for the generalized hypergeometric function of one variable, ${}_AF_B$ (see [15]),

$${}_AF_B\left(\left.\begin{array}{c}a_1,\ \ldots\ ,a_A\\ b_1,\ \ldots\ ,b_B\end{array}\right| z\right)=\sum_{j=0}^{\infty}\frac{(a_1)_j\ \ldots\ (a_A)_j}{(b_1)_j\ \ldots\ (b_B)_j}\,\frac{z^j}{j!}\,, \tag{5}$$

where $(a)_j\equiv\Gamma(a+j)/\Gamma(a)$ is the Pochhammer symbol. In particular, if $\nu_1=\nu_2=1$ we obtain from (4) (after expanding in $\varepsilon=(4-n)/2$) the well-known result in terms of elementary functions (see, e.g., [1, 16]). If we close the interation contour in (3) to the left we obtain the sum of three functions ${}_3F_2$ of the variable $4m^2/k^2$ (see [7]) which corresponds to the analytic continuation of the function ${}_3F_2$ from the variable z to $1/z$.

It should be also noted that in the paper [7] the results for two-point integrals with different masses and for some triangle diagrams are also presented. In refs. [8, 9] general results for massive one-loop diagrams with an arbitrary number of external lines are obtained.

3. In this section we shall consider some examples of evaluating two-loop diagrams. Let us consider first the diagram in Fig. 2 where massive particles (quarks) are denoted by continuous lines, whereas wavy lines correspond to the massless particles (gluons). The corresponding Feynman integral is of the following form:

$$L(\nu,\rho,\sigma;m) \equiv \int\int \frac{\mathrm{d}^n p\, \mathrm{d}^n q}{(p^2-m^2)^\nu\left((k-q)^2-m^2\right)^\rho\left((q-p)^2\right)^\sigma}\ . \tag{6}$$

Note that in real quark-gluon interactions such diagrams are absent (since in the QCD Lagrangian we have no quadric vertex "quark-quark-gluon-gluon"). Nevertheless, below we shall see that some more complicated two-loop diagrams can be reduced to such diagrams by use of the integration-by-parts technique [11, 12].

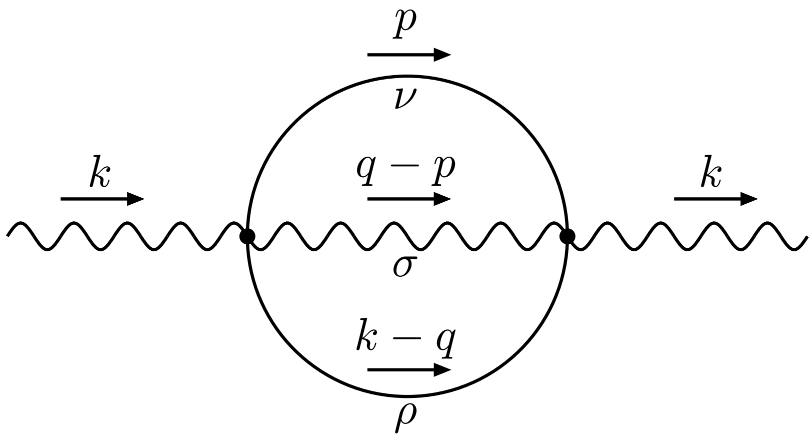


Fig. 2

Using the technique [7]–[9] we obtain the following Mellin–Barnes representation for the integral (6):

$$\begin{aligned} L(\nu,\rho,\sigma;m) &= \pi^n \mathrm{i}^{2-2n}(-m^2)^{n-\nu-\rho-\sigma}\, \frac{\Gamma(n/2-\sigma)}{\Gamma(\nu)\Gamma(\rho)\Gamma(\sigma)}\, \frac{1}{2\pi\mathrm{i}} \int\limits_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}u \left(-\frac{k^2}{m^2}\right)^u \Gamma(-u) \\ &\times \frac{\Gamma(\sigma+u)\Gamma(\nu+\sigma-n/2+u)\Gamma(\rho+\sigma-n/2+u)\Gamma(\nu+\rho+\sigma-n+u)}{\Gamma(\nu+\rho+2\sigma-n+2u)\Gamma(n/2+u)}\ . \end{aligned} \tag{7}$$

Closing the integration contour to the right, we find

$$\begin{aligned} L(\nu,\rho,\sigma;m) &= \pi^n \mathrm{i}^{2-2n}(-m^2)^{n-\nu-\rho-\sigma} \\ &\times \frac{\Gamma(n/2-\sigma)\Gamma(\nu+\sigma-n/2)\Gamma(\rho+\sigma-n/2)\Gamma(\nu+\rho+\sigma-n)}{\Gamma(\nu)\Gamma(\rho)\Gamma(\nu+\rho+2\sigma-n)\Gamma(n/2)} \\ &\times {}_4F_3\left(\begin{array}{c} \sigma,\ \nu+\sigma-n/2,\ \rho+\sigma-n/2,\ \nu+\rho+\sigma-n \\ \sigma+(\nu+\rho-n)/2,\ \sigma+(\nu+\rho-n+1)/2,\ n/2 \end{array} \middle| \frac{k^2}{4m^2} \right), \end{aligned} \tag{8}$$

where the hypergeometric function ${}_4F_3$ is a particular case of the formula (5). If $k^2 = 0$ then ${}_4F_3 = 1$, and we obtain the well-known result for vacuum two-loop diagram. As a particular case, we can also present result for the convergent integral with $\nu = \rho = 2$, $\sigma = 1$:

$$L(2,2,1;m) = \pi^4 m^{-2}\, {}_3F_2\left(\begin{array}{c} 1,1,1 \\ 2,3/2 \end{array} \middle| \frac{k^2}{4m^2} \right) , \tag{9}$$

where (see in [17])

$${}_3F_2\left(\begin{array}{c} 1,1,1 \\ 2,3/2 \end{array} \middle| z \right) = \begin{cases} z^{-1} \arcsin^2 \sqrt{z}, & z \geq 0 \\ -z^{-1} \ln^2 \left(\sqrt{1-z} + \sqrt{-z} \right), & z \leq 0 \end{cases}$$

It should be also noted that if we close the contour in (9) to the left then we obtain the sum of four functions ${}_4F_3$ of the variable $4m^2/k^2$ (this corresponds to the analytic continuation of the function ${}_4F_3$ from z to $1/z$).

4. The evaluation of two-loop diagrams of more general form (see Fig. 3) is very interesting in various examinations. For example, such diagrams contribute to the gluon polarization operator. We are also confronted with such diagrams when studying meson parameters by applying the QCD sum rules method (see, e.g., in [18]). The corresponding Feynman integral is defined by

$$V(\nu_1, \nu_2, \rho_1, \rho_2, \sigma; m) \equiv \int\!\!\int \frac{\mathrm{d}^n p \, \mathrm{d}^n q}{(p^2 - m^2)^{\nu_1} \left((k-p)^2 - m^2\right)^{\nu_2} (q^2 - m^2)^{\rho_1} \left((k-q)^2 - m^2\right)^{\rho_2} \left((q-p)^2\right)^{\sigma}} . \tag{10}$$

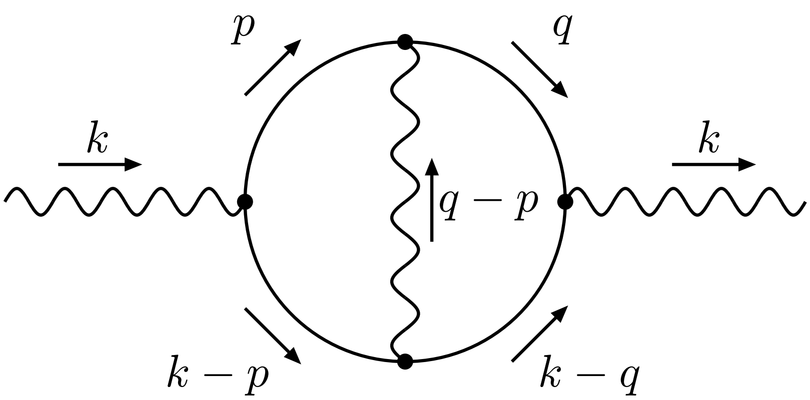


Fig. 3

A straightforward application of the considered technique to the integrals of the type (10) with arbitrary values of the powers of denominators does not make it possible to obtain a general result in the form of known hypergeometric functions. However, in realistic calculations we are interested in integrals with integer values of the powers of denominators. We shall show that integration by parts makes it possible to reduce the integrals (10) with integer powers of denominators to known integrals (2) and (6).

One can easily see that the main identity of the integration-by-parts method [11, 12] does not change its form when we pass from the massless integrals of the type (10) to the massive ones:

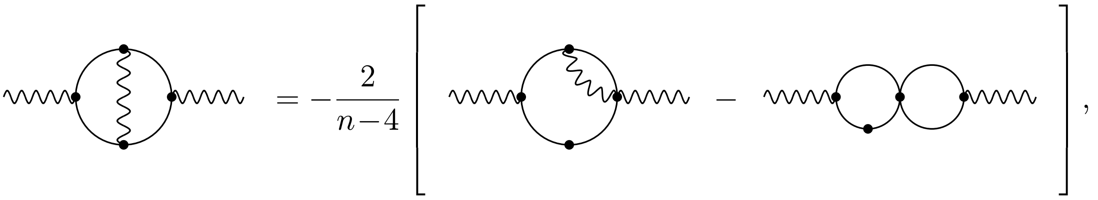
$$\begin{aligned}(n-\rho_1-\rho_2-2\sigma)\ &V(\nu_1,\nu_2,\rho_1,\rho_2,\sigma;m)\\ &+\rho_1 V(\nu_1-1,\nu_2,\rho_1+1,\rho_2,\sigma;m)-\rho_1 V(\nu_1,\nu_2,\rho_1+1,\rho_2,\sigma-1;m)\\ &+\rho_2 V(\nu_1,\nu_2-1,\rho_1,\rho_2+1,\sigma;m)-\rho_2 V(\nu_1,\nu_2,\rho_1,\rho_2+1,\sigma-1;m)=0\ .\end{aligned} \tag{11}$$

As for the massless case, it is sufficient to use this identity and obvious symmetry conditions,

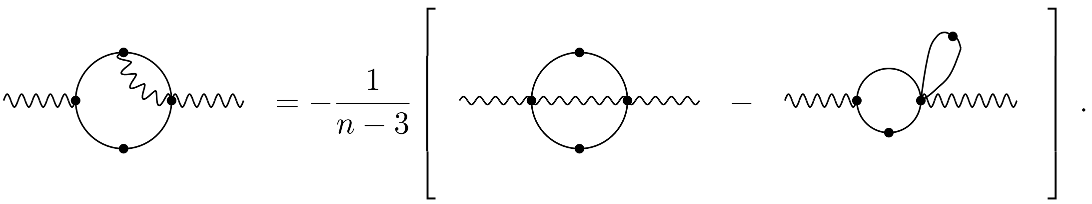
$$V(\nu_1,\nu_2,\rho_1,\rho_2,\sigma;m)=V(\rho_1,\rho_2,\nu_1,\nu_2,\sigma;m)=V(\nu_2,\nu_1,\rho_2,\rho_1,\sigma;m)\ , \tag{12}$$

to calculate all integrals with integer values of ν_1, ν_2, ρ_1, ρ_2, σ.

As an example, let us consider the integral with $\nu_1=\nu_2=\rho_1=\rho_2=\sigma=1$. By analogy with [11], we obtain from (11) the following diagrammatic equation:


(13)

where a dot on the line denotes the appearance of the corresponding denominator in the second power. The second term in the square brackets can be repesented as a product of two integrals of the type (2). However, in contrast to the massless case (when the result for the first diagram in square brackets is well known), the massive case requires the second use of the identity [11]:


(14)

The second term in square brackets can also be represented as a product of two one-loop integrals (2), while the first term corresponds to the two-loop integral (6). Finally we have

$$\begin{aligned}V(1,1,1,1,1;m)\ &=\ \frac{2}{(n-3)(n-4)}\{L(2,2,1;m)\\ &\quad +J(2,1;m)\,[(n-3)J(1,1;m)-J(2,0;m)]\}\ .\end{aligned} \tag{15}$$

It should be noted that the singularity in $\varepsilon=(4-n)/2$ in the denominator implies that all integrals in the square brackets should be evaluated keeping terms of the order ε. The results (4) and (8) allow to do this since n is arbitrary in these formulae. It should also be noted that the result for $V(1,1,1,1,1;m)$ obtained in this way coincides with that in the paper [5].

5. Thus, in this paper we have presented some results for loop diagrams obtained by use of the method [7]–[9]. We have also shown that some more complicated two-loop diagrams can be evaluated by using this method in combination with the integration by parts [11, 12].

The presented approach is rather useful for the algorithmization of calculations, especially if we are interested only in a restricted number of asymptotic expansion terms (for example, the expansion in k^2/m^2). Numerators and denominators of the expansion coefficients of hypergeometric functions can be presented in terms of Pochhammer symbols (see in [5]). When considering the limits $\varepsilon \to 0$ and $\nu_i \to$ integers, some combinations of ψ functions can also appear in the numerators. Such expansions correspond to the derivatives of hypergeometric functions with respect to parameters. For the integrals (10), the coefficients of the expansion in k^2/m^2 are rational numbers.

It should be noted that, when studying expansions in m^2/k^2, as a rule, we are confronted with the logarithmic case of analytic continuation of corresponding hypergeometric functions of k^2/m^2 (see, e.g., in [15]). As a result, in the region of small values of m^2/k^2 expressions for the integrals (10) have terms with $m^2 \ln(-m^2/k^2)$ and $m^2 \ln^2(-m^2/k^2)$ (see, e.g., in [5]). The limit $m \to 0$ gives the well-known result for the integral (15): $-\pi^4 \, 6\zeta(3)/k^2$.

We hope that the examined algorithms will be also useful in more complicated multiloop calculations.

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